

# Weak solutions of fractional differential equations in non cylindrical domains

A. Kubica<sup>1</sup>, P. Rybka<sup>2</sup>, K. Ryszewska<sup>1</sup>

<sup>1</sup> Department of Mathematics and Information Sciences  
Warsaw University of Technology  
ul. Koszykowa 75, 00-662 Warsaw, Poland  
e-mail: A.Kubica@mini.pw.edu.pl

<sup>2</sup> Institute of Applied Mathematics and Mechanics, Warsaw University  
ul. Banacha 2, 02-097 Warsaw, Poland  
e-mail: rybka@mimuw.edu.pl

August 5, 2016

**Abstract.** We study a time fractional heat equation in a non cylindrical domain. The problem is one-dimensional. We prove existence of properly defined weak solutions by means of the Galerkin approximation.

**Key words:** time fractional Caputo derivative, heat equation, Galerkin method

**2010 Mathematics Subject Classification.** Primary: 35R11 Secondary: 35K45

## 1 Introduction

### 1.1 Motivation

In this paper we study the heat equation with the Caputo time derivative in a non cylindrical domain. It is intended to be the first in a series devoted the Stefan problem with fractional derivatives. We address here a simple, yet non-trivial question of the existence of solutions to the problem where the interfacial curve is given to us.

The interest in fractional PDE's stems from many sources. One of them is the theory of stochastic processes admitting jumps and continuous paths, see [1], [2], [7]. Our motivation comes from phenomenological models of sediment transport, see [5], [10], [11], [9]. Apparently, this problem awaits a systematic treatment.

In these problems, the position of the advancing front  $s$  is not known, i.e., it is a part of the problem. Here, we consider a simplified situation,

$$\begin{cases} D_s^\alpha u(x, t) = u_{xx}(x, t) + f(x, t) & \text{for } 0 < x < s(t), \ 0 < t < T, \\ -u_x(0, t) = h(t) & \text{for } 0 < t < T, \\ u(s(t), t) = 0 & \text{for } 0 < t < T, \\ u(x, 0) = u_0(x) & \text{for } 0 < x < b = s(0), \end{cases} \quad (1.1)$$

where  $h(t)$ ,  $f(x, t)$ ,  $u_0(x)$ ,  $s$  are given functions and  $s$  is nondecreasing. By  $D_s^\alpha$  we understand the Caputo fractional derivative, defined as follows

$$D_s^\alpha w(x, t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} w_t(x, \tau) d\tau & \text{for } x \leq s(0), \\ \frac{1}{\Gamma(1-\alpha)} \int_{s^{-1}(x)}^t (t-\tau)^{-\alpha} w_t(x, \tau) d\tau & \text{for } x > s(0). \end{cases} \quad (1.2)$$

In the original free boundary problem the above system is augmented by an equation governing the evolution of  $s$ , here  $s$  is given to us. We assume that  $s$  is increasing. We will also use the following convention: if  $x > b$ , then  $s^{-1}(x) = \max\{t : s(t) = x\}$  and if  $x \in [0, b]$ , then  $s^{-1}(x) = 0$ .

Experience with fractional derivatives tells us that tools used for PDE's require substantial modification before they can be applied to equations with fractional derivatives. In case of time fractional derivatives this means that we have to take into account the whole history of the process. This is the main difficulty of the analysis. This is why we set limited goals in this paper.

Our main result, expressed in Theorem 1.1 below, is the existence of properly defined weak solutions to (1.1). We use for this purpose the Galerkin method. This is a straightforward approach in case of PDE's, however, here it gets complicated when applied to (1.1). Now, we briefly describe the content of our paper. In Section 2 we construct approximate solutions by means of the Galerkin method. In Section 3 we derive the necessary estimate and show that we can pass to the limit. This process yields a existence of a weak solution. Proofs of technical results are presented in the Appendix.

## 1.2 Preliminaries and a definition of a weak solution

Here, we will make our assumptions, upheld throughout this paper. We will denote by  $\alpha \in (0, 1)$  the order of the Caputo derivative. By  $s$  we denote the position of the interface. Its initial position  $b = s(0)$  is positive. Function  $s$  is not only increasing and continuous, but also

$$t \mapsto t^{1-\alpha} \dot{s}(t) \in C([0, T]) \quad \text{and} \quad \dot{s} \geq 0. \quad (1.3)$$

With the help of  $s$ , we define a non cylindrical domain  $Q_{s,t}$  by the following formula,

$$Q_{s,t} = \{(x, \tau) : 0 < x < s(\tau), \ 0 < \tau < t\}.$$

The fractional derivative  $D_s^\alpha$  is a complicated operator. In order to simplify the analysis, we introduce an auxiliary integral operator for functions defined on the domain  $Q_{s,t}$

by the following formula,

$$I_s^\alpha w(x, t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} w(x, \tau) d\tau & \text{for } x \leq s(0), \\ \frac{1}{\Gamma(\alpha)} \int_{s^{-1}(x)}^t (t - \tau)^{\alpha-1} w(x, \tau) d\tau & \text{for } x > s(0). \end{cases} \quad (1.4)$$

It is easy to verify that if  $u(x, t)$  is an absolutely continuous function with respect to  $t$  variable and it satisfies the boundary and initial conditions of (1.1), then the following identity holds

$$D_s^\alpha u(x, t) = \frac{d}{dt} I_s^{1-\alpha} [u(x, t) - \tilde{u}_0(x)], \quad (1.5)$$

where  $\tilde{u}_0(x)$  will always denote the extension of  $u_0(x)$  by zero. This equality suggests a weak form of the system (1.1).

**Definition 1.1.** Let us assume that  $f \in L^2(Q_{s,T})$ ,  $h \in C^1([0, T])$  and  $u_0 \in L^2(0, b)$ . A function  $u$  is a *weak solution* of (1.1) if  $u, u_x \in L^2(Q_{s,T})$ ,  $I^{1-\alpha} \tilde{u}(\cdot, t) \in L^\infty(0, T; L^2(0, s(T)))$  and  $u$  fulfills the identity

$$\begin{aligned} & - \int_{Q_{s,T}} I_s^{1-\alpha} [u(x, t) - \tilde{u}_0(x)] \varphi_t(x, t) dx dt + \int_{Q_{s,T}} u_x(x, t) \varphi_x(x, t) dx dt \\ & = \int_0^T h(t) \varphi(0, t) dt + \int_{Q_{s,T}} f(x, t) \varphi(x, t) dx dt \end{aligned} \quad (1.6)$$

for all  $\varphi \in C^1(\overline{Q_{s,T}})$  such that  $\varphi(x, T) = 0$  for  $x \in [0, s(T)]$  and  $\varphi(s(t), t) = 0$  for  $t \in [0, T]$ .

Now we are ready to formulate our main result.

**Theorem 1.1.** Let us assume that  $\alpha \in (0, 1)$ ,  $b > 0$  and  $s$  satisfies (1.3). If  $f \in L^2(Q_{s,T})$ ,  $h \in C^1([0, T])$  and  $u_0 \in L^2(0, b)$  then, there exists a weak solution of (1.1).

The proof will be provided in Section 3. In its first step we reduce the problem to the case of zero boundary condition of (1.1). For this purpose we fix a smooth function  $\eta = \eta(x)$  such that  $\eta'(0) = 1$  and  $\eta(x) = 0$  for  $x \geq b$ . Then  $v(x, t) := u(x, t) - h(t)\eta(x)$  satisfies

$$\begin{cases} D_s^\alpha v(x, t) = v_{xx}(x, t) + g(x, t) & \text{in } Q_{s,T}, \\ v_x(0, t) = 0 & \text{for } 0 < t < T, \\ v(s(t), t) = 0 & \text{for } 0 < t < T, \\ v(x, 0) = v_0(x) & \text{for } 0 < x < b, \end{cases} \quad (1.7)$$

where

$$g(x, t) = f(x, t) + h(t)\eta_{xx}(x) - \eta(x)D^\alpha h(t), \quad v_0 = u_0 - h(0)\eta(x). \quad (1.8)$$

Therefore, we shall show that there exists  $v(x, t)$  such that the identity

$$- \int_{Q_{s,T}} I_s^{1-\alpha} [v(x, t) - \tilde{v}_0(x)] \varphi_t(x, t) dx dt + \int_{Q_{s,T}} v_x(x, t) \varphi_x(x, t) dx dt = \int_{Q_{s,T}} g(x, t) \varphi(x, t) dx dt, \quad (1.9)$$

holds for each  $\varphi$  as in the definition 1.1, where  $\tilde{v}_0(x)$  is a continuation by zero of  $v_0(x)$ .

The proof of this Theorem is based on the Galerkin method. In Section 2 we construct an approximate solution. In Section 3 we pass to the limit, after having derived necessary a priori estimates in Lemmas 3.1 and 3.2. Many auxiliary facts used in our analysis are collected in the Appendix.

## 2 Approximate solutions

We are looking for approximate solutions of (1.9) in the following form

$$v_m(x, t) = \sum_{n=0}^m c_{n,m}(t) \varphi_n(x, t), \quad (2.1)$$

where for each  $t \in [0, T]$  functions  $\{\varphi_n(\cdot, t)\}_{n \in \mathbb{N}}$  constitute an orthonormal basis in  $L^2(0, s(t))$ . More precisely, we define functions  $\varphi_n$  as follows:

$$-\varphi_{n,xx}(x, t) = \lambda_n^2(t) \varphi_n(x, t), \quad \varphi_{n,x}(0, t) = 0, \quad \varphi_n(s(t), t) = 0. \quad (2.2)$$

These conditions imply that  $\varphi_n(x, t) = \sqrt{\frac{2}{s(t)}} \cos \lambda_n(t)x$ , where

$$\lambda_n(t) = \frac{\pi}{s(t)} \left[ \frac{1}{2} + n \right]. \quad (2.3)$$

We also have

$$\int_0^{s(t)} \varphi_n(x, t) \cdot \varphi_k(x, t) dx = \delta_{k,n}, \quad (2.4)$$

$$\int_0^{s(t)} \varphi_{n,x}(x, t) \cdot \varphi_{k,x}(x, t) dx = \lambda_n^2(t) \delta_{k,n}. \quad (2.5)$$

Subsequently, we define vector functions  $c_m(t) = (c_{0,m}(t), \dots, c_{m,m}(t))$  as solutions to the following system

$$\int_0^{s(t)} D_s^\alpha v_m(x, t) \cdot \varphi_k(x, t) dx + \int_0^{s(t)} v_{m,x}(x, t) \cdot \varphi_{k,x}(x, t) dx = \int_0^{s(t)} g_m^\varepsilon(x, t) \varphi_k(x, t) dx, \quad t \in (0, T] \quad (2.6)$$

$$c_{k,m}(0) = \int_0^b v_0(x) \varphi_k(x, 0) dx, \quad k = 0, \dots, m, \quad (2.7)$$

where

$$g_m^\varepsilon(x, t) = \sum_{n=0}^m \int_0^{s(t)} g^\varepsilon(y, t) \varphi_n(y, t) dy \varphi_n(x, t).$$

We assume that  $g^\varepsilon(x, t)$  is a smooth (with respect to time variable) approximation of  $g(x, t)$  obtained by convolution of mollifier and  $\tilde{g}$ . In this section we shall prove the following result.

**Theorem 2.1.** *Let us assume that  $s$  satisfies (1.3) and  $\alpha \in (0, 1)$ ,  $b > 0$ ,  $T > 0$ , in addition  $f \in L^2(Q_{s,T})$ ,  $h \in C^1([0, T])$  and  $u_0 \in L^2(0, b)$ . Then, for each  $m \in \mathbb{N}$  there exists  $c_m(\cdot) \in AC([0, T]; \mathbb{R}^m)$ , satisfying (2.7), such that  $v_m$  given by (2.1) satisfies (2.6). Furthermore,  $t^{1-\alpha}c'_m \in C([0, T]; \mathbb{R}^m)$  and  $t^{1-\alpha}v_{m,t}$  is continuous on  $\overline{Q_{s,T}}$ .*

**Remark 2.1.** System (2.6) can be written in a form suitable for  $c_m(t)$ , which are only continuous. Then, the proof of the existence of continuous solutions  $c_m(t)$  is much simpler than presented below because it is sufficient to assume absolute continuity of  $s$ . However, in this case, we are not able to deduce absolute continuity of solutions  $c_m(t)$ . As a result, we are not able to obtain a priori estimates for an approximate solution (see Lemma 3.1). In order to obtain them, we were forced to assume that  $t^{1-\alpha}\dot{s}(t)$  is continuous and we are looking for  $c_m(t)$  in a special function space.

The proof of Theorem 2.1 will be divided into several steps, each is contained in a separate subsection. At the first stage, see §2.1, we set up the auxiliary integral equation for the appropriate solutions. Next, in §2.2 we find a suitable function space. In §2.3 we show existence of local in time solution to the system introduced in §2.1. Finally, in §2.4 we show its global solvability.

**Notation** We denote by  $c_0$  a generic constant depending on  $b, m, \alpha, T, \max\{|s(t)| : t \in [0, T]\}$  and  $\max\{t^{1-\alpha}\dot{s}(t) : t \in [0, T]\}$ .

## 2.1 Reformulation of system (2.6)

In order to streamline the argument and to make it transparent, we moved a number of estimates to the Appendix. We also recall there the definition of the Euler Beta function, (4.2), and its relation  $\Gamma$  function, (4.3)

We note that once we assume (1.3), then for each  $x$  functions  $\varphi_n(x, \cdot)$  are absolutely continuous. Under the assumption that  $c_m$  are absolutely continuous, the system (2.6) with  $v_m$  given by (2.1) can be equivalently written as follows,

$$\sum_{n=0}^m \int_0^{s(t)} D_s^\alpha [c_{n,m}(t) \varphi_n(x, t)] \cdot \varphi_k(x, t) dx + \lambda_k^2(t) c_{k,m}(t) = \int_0^{s(t)} g^\varepsilon(x, t) \varphi_k(x, t) dx,$$

where we used (2.4) and (2.5). Thus,

$$\begin{aligned} & \sum_{n=0}^m \int_0^t \int_0^{s(\tau)} (t - \tau)^{-\alpha} c'_{n,m}(\tau) \cdot \varphi_n(x, \tau) \cdot \varphi_k(x, t) dx d\tau \\ & + \sum_{n=0}^m \int_0^t \int_0^{s(\tau)} (t - \tau)^{-\alpha} c_{n,m}(\tau) \cdot (\varphi_n(x, \tau))_\tau \cdot \varphi_k(x, t) dx d\tau \\ & + \Gamma(1 - \alpha) \lambda_k^2(t) c_{k,m}(t) = \Gamma(1 - \alpha) \int_0^{s(t)} g^\varepsilon(x, t) \varphi_k(x, t) dx. \end{aligned}$$

Hence, making use of (2.4), we obtain

$$\begin{aligned}
& \int_0^t (t-\tau)^{-\alpha} c'_{k,m}(\tau) d\tau + \sum_{n=0}^m \int_0^t (t-\tau)^{-\alpha} c'_{n,m}(\tau) \int_0^{s(\tau)} \varphi_n(x, \tau) [\varphi_k(x, t) - \varphi_k(x, \tau)] dx d\tau \\
& + \sum_{n=0}^m \int_0^t (t-\tau)^{-\alpha} c_{n,m}(\tau) \int_0^{s(\tau)} (\varphi_n(x, \tau))_\tau \cdot \varphi_k(x, t) dx d\tau \\
& + \Gamma(1-\alpha) \lambda_k^2(t) c_{k,m}(t) = \Gamma(1-\alpha) \int_0^{s(t)} g^\varepsilon(x, t) \varphi_k(x, t) dx.
\end{aligned}$$

In order to make further computation more transparent, we introduce the following notation,

$$\begin{aligned}
B(\tau, t)_{n,k} &= \int_0^{s(\tau)} \varphi_n(x, \tau) [\varphi_k(x, t) - \varphi_k(x, \tau)] dx \quad \text{for } 0 \leq \tau \leq t \leq T, \\
D(\tau, t)_{n,k} &= \int_0^{s(\tau)} (\varphi_n(x, \tau))_\tau \cdot \varphi_k(x, t) dx \quad \text{for } 0 \leq \tau \leq t \leq T, \\
E(t)_{n,k} &= \Gamma(1-\alpha) \lambda_k^2(t) \delta_{n,k} \quad \text{for } 0 \leq t \leq T, \\
G^\varepsilon(t)_k &= \Gamma(1-\alpha) \int_0^{s(t)} g^\varepsilon(x, t) \varphi_k(x, t) dx \quad \text{for } 0 \leq t \leq T.
\end{aligned}$$

With this notation system (2.6) may be written in a more compact way as follows,

$$\int_0^t (t-\tau)^{-\alpha} c'_m(\tau) d\tau + \int_0^t (t-\tau)^{-\alpha} B(\tau, t) c'_m(\tau) d\tau + \int_0^t (t-\tau)^{-\alpha} D(\tau, t) c_m(\tau) d\tau + E(t) c_m(t) = G^\varepsilon(t). \quad (2.8)$$

After having applied  $\Gamma(\alpha) I^\alpha$  to both sides of (2.8) and using (4.4), we arrive at

$$\begin{aligned}
c_m(t) &= c_m(0) - c_\alpha \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) c'_m(p) dp d\tau \\
& - c_\alpha \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} D(p, \tau) c_m(p) dp d\tau - c_\alpha \int_0^t (t-\tau)^{\alpha-1} E(\tau) c_m(\tau) d\tau + \frac{1}{\Gamma(1-\alpha)} I^\alpha G^\varepsilon(t),
\end{aligned} \quad (2.9)$$

where

$$c_\alpha = \frac{1}{\Gamma(\alpha) \Gamma(1-\alpha)}. \quad (2.10)$$

We decompose function  $D$ , which will allow us to obtain useful estimates.

$$D(\tau, t)_{n,k} = \int_0^{s(\tau)} (\varphi_n(x, \tau))_\tau \cdot \varphi_k(x, \tau) dx + \int_0^{s(\tau)} (\varphi_n(x, \tau))_\tau \cdot [\varphi_k(x, t) - \varphi_k(x, \tau)] dx.$$

Using (2.2) and (2.4), we have

$$\int_0^{s(\tau)} (\varphi_n(x, \tau))_\tau \cdot \varphi_n(x, \tau) dx = \frac{1}{2} \left[ \underbrace{\int_0^{s(\tau)} \varphi_n^2(x, \tau) dx}_{=1} \right]_\tau - \frac{\dot{s}(\tau)}{2} \underbrace{\varphi_n^2(s(\tau), \tau)}_{=0} = 0.$$

For  $k \neq n$ , we get

$$\int_0^{s(\tau)} (\varphi_n(x, \tau))_\tau \cdot \varphi_k(x, \tau) dx = \frac{\dot{s}(\tau)}{s(\tau)} \hat{D}_{n,k},$$

for a constant matrix  $\hat{D}$ . Therefore, we have

$$D(\tau, t)_{n,k} = \frac{\dot{s}(\tau)}{s(\tau)} \hat{D}_{n,k} + \tilde{D}(\tau, t)_{n,k}, \quad (2.11)$$

where

$$\tilde{D}(\tau, t)_{n,k} = \int_0^{s(\tau)} (\varphi_n(x, \tau))_\tau \cdot [\varphi_k(x, t) - \varphi_k(x, \tau)] dx. \quad (2.12)$$

Taking into account (2.11) and the properties of the Beta function, we can write system (2.9) in this way

$$\begin{aligned} c_m(t) = & c_m(0) - c_\alpha \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) c'_m(p) dp d\tau - \int_0^t \frac{\dot{s}(p)}{s(p)} \hat{D} c_m(p) dp \\ & - c_\alpha \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) c_m(p) dp d\tau - c_\alpha \int_0^t (t-\tau)^{\alpha-1} E(\tau) c_m(\tau) d\tau + \frac{1}{\Gamma(1-\alpha)} I^\alpha G^\varepsilon(t). \end{aligned} \quad (2.13)$$

Our goal is to show that integral equation (2.13) has an absolutely continuous solution  $c_m$ , such that  $t^{1-\alpha} c'_m(t)$  is continuous.

**Remark 2.2.** If  $s$  is constant, i.e.  $s(t) \equiv b$  (in this case  $Q_{s,T}$  is a cylindrical domain), then equation (2.13) significantly simplifies, because in this situation  $B \equiv 0$ ,  $\tilde{D} \equiv 0$ ,  $\dot{s} = 0$  and  $E$  is a constant matrix.

We shall prove the existence of a solution of (2.13) by applying the Banach fixed point theorem on an appropriate space. For this purpose we collect here estimates of  $B$ ,  $\tilde{D}$  and  $E$ , which we will use later on.

**Corollary 2.1.** *Let us assume that  $s \in AC([0, T])$ ,  $\dot{s} \geq 0$  and  $s(0) = b$ , then*

$$|B(\tau, t)| \leq c_0[s(t) - s(\tau)], \quad (2.14)$$

$$|B_\tau(\tau, t)| \leq c_0 \dot{s}(\tau), \quad (2.15)$$

$$|B_t(\tau, t)| \leq c_0 \dot{s}(t), \quad (2.16)$$

$$|B(\tau, t_2) - B(\tau, t_1)| \leq c_0 |s(t_2) - s(t_1)|, \quad (2.17)$$

$$|\tilde{D}(\tau, t)| \leq c_0 \dot{s}(\tau)[s(t) - s(\tau)], \quad (2.18)$$

$$|\tilde{D}_t(\tau, t)| \leq c_0 \dot{s}(\tau) \dot{s}(t), \quad (2.19)$$

$$|E'(t)| \leq c_0 \dot{s}(t), \quad (2.20)$$

hold a.e., where  $c_0$  depends only on  $m, b, \alpha, \|t^{1-\alpha} \dot{s}\|_{C[0,T]}$  and  $T$ .

*Proof.* We reach these conclusions by simple calculations.  $\square$

**Corollary 2.2.** *We assume that  $s \in AC([0, T])$ ,  $s(0) = b$  and condition (1.3) holds, then*

$$|s(\tau) - s(p)| \leq c_0 |\tau^\alpha - p^\alpha| \leq c_0 |\tau - p|^\alpha, \quad (2.21)$$

$$|s(\tau) - s(p)| \leq c_0 p^{\alpha-1} |\tau - p|, \quad \text{for } p < \tau. \quad (2.22)$$

*Proof.* Also in this case by inspection we establish validity of these statements.  $\square$

We draw further conclusions from Corollaries 2.1 and 2.2. Namely, (2.14), (2.18) and (2.22) imply that

$$\lim_{t \rightarrow \tau^+} (t - \tau)^{-\alpha} B(\tau, t) = 0 \quad \text{for } \tau > 0, \quad (2.23)$$

$$\lim_{t \rightarrow \tau^+} (t - \tau)^{-\alpha} \tilde{D}(\tau, t) = 0 \quad \text{for } \tau > 0. \quad (2.24)$$

## 2.2 Our choice of the function space

After these preparations we define a function space, where we shall look for a solution to (2.13). For  $T > 0$  and  $c_m(0)$  as in (2.7) we set,

$$X(T) = \{c \in C^1((0, T]; \mathbb{R}^m) : c(0) = c_m(0), t^{1-\alpha} c'(t) \in C([0, T]; \mathbb{R}^m)\}. \quad (2.25)$$

It is easy to check that this space endowed with the natural norm

$$\|c\|_{X(T)} = \|c\|_{C([0, T])} + \|t^{1-\alpha} c'\|_{C([0, T])}$$

is a Banach space.

**Remark 2.3.** Having introduced the norm of  $X(T)$ , we may write that by  $c_0$  we denote a generic constant  $c_0 = c_0(b, m, \alpha, T, \|s\|_{X(T)})$ .

For each  $c \in X(T)$ , we define a function

$$(Pc)(t) = c_m(0) - (P_1 c)(t) - (P_2 c)(t) - (P_3 c)(t) - (P_4 c)(t) + \frac{1}{\Gamma(1-\alpha)} I^\alpha G^\varepsilon(t),$$

where

$$(P_1 c)(t) = c_\alpha \int_0^t (t - \tau)^{\alpha-1} \int_0^\tau (\tau - p)^{-\alpha} B(p, \tau) c'(p) dp d\tau,$$

$$(P_2 c)(t) = \int_0^t \frac{\dot{s}(p)}{s(p)} \hat{D}c(p) dp,$$



$$(P_3c)(t) = c_\alpha \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) c(p) dp d\tau,$$

$$(P_4c)(t) = c_\alpha \int_0^t (t-\tau)^{\alpha-1} E(\tau) c(\tau) d\tau$$

and  $c_\alpha$  is given by (2.10). With this definition of  $P$ , we can concisely write (2.13) as

$$c_m(t) = c_m(0) + (Pc_m)(t), \quad t \in [0, T]. \quad (2.26)$$

We shall see that  $P$  maps  $X(T)$  into itself. This is done in Lemma 2.3 below. Then, in next subsection, we shall show that for a sufficiently small  $T$  operator  $P$  will be a contraction. Thus, by Banach Theorem, operator  $P$  has a unique fixed point.

In order to proceed, we state estimates frequently used to establish Lemma 2.3. They are shown in the Appendix.

**Lemma 2.1.** *If  $f \in AC[0, T]$ , then  $(I^\alpha f)(t) \in AC[0, T]$  and  $(I^\alpha f)'(t) = I^\alpha f'(t) + t^{\alpha-1} \frac{f(0)}{\Gamma(\alpha)}$ .*

**Lemma 2.2.** *Let us assume that  $p \mapsto p^{1-\alpha} w(p) \in L^\infty(0, T)$  and let*

$$g_2(\tau) = \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) w(p) dp.$$

*Then, there exists an absolutely continuous representative of integrable function  $g_2$ . That is, there exists  $\tilde{g}_2 \in AC[0, T]$  satisfying  $\tilde{g}_2(0) = 0$  and such that  $g_2 = \tilde{g}_2$  a.e. on  $[0, T]$ .*

Now, we are ready for the main result of this subsection.

**Lemma 2.3.** *Let us take any  $T > 0$  and  $c \in X(T)$ , then  $Pc \in X(T)$ .*

*Proof.* If  $c \in X(T)$ , then we set

$$\mathcal{P}_i(t) = t^{1-\alpha} (P_i c)'(t), \quad i = 1, \dots, 4.$$

At first, we will show that

$$\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4 \in C([0, T]; \mathbb{R}^m). \quad (2.27)$$

We consider each  $\mathcal{P}_i$ ,  $i = 1, \dots, 4$  separately. However, we first record for all elements  $c \in X(T)$  the following bound

$$|c'(p)| \leq \|c\|_{X(T)} p^{\alpha-1}. \quad (2.28)$$

**Case of  $\mathcal{P}_1$ .** Our goal is to prove that

$$\mathcal{P}_1 \in C([0, T]; \mathbb{R}^m). \quad (2.29)$$

If we keep in mind Lemma 2.1 and Lemma 2.2, then it is enough to show that

$$t \mapsto \mathcal{P}_{1,1}(t) := t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) c'(p) dp \right]_\tau d\tau \in C([0, T]; \mathbb{R}^m). \quad (2.30)$$

In order to prove (2.30), we consider the difference, where  $t_1 < t_2$ ,

$$\begin{aligned} & t_2^{1-\alpha} \int_0^{t_2} (t_2-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) c'(p) dp \right]_\tau d\tau \\ & - t_1^{1-\alpha} \int_0^{t_1} (t_1-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) c'(p) dp \right]_\tau d\tau \\ = & t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) c'(p) dp \right]_\tau d\tau \\ & + t_2^{1-\alpha} \int_0^{t_1} [(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1}] \left[ \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) c'(p) dp \right]_\tau d\tau \\ & + (t_2^{1-\alpha} - t_1^{1-\alpha}) \int_0^{t_1} (t_1-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) c'(p) dp \right]_\tau d\tau \\ \equiv & W_1(t_1, t_2) + W_2(t_1, t_2) + W_3(t_1, t_2). \end{aligned}$$

By definition, in the case of  $t_1 = 0$  we have  $W_2(0, t_2) = 0$  and  $W_3(0, t_2) = 0$ , hence continuity of  $\mathcal{P}_{1,1}$  at  $t = 0$  will be shown if  $\lim_{t_2 \rightarrow 0} W_1(0, t_2) = 0$ . For this purpose, we estimate  $W_1$ . With the help of (2.23) we perform the differentiation with respect to  $\tau$  to get,

$$\begin{aligned} W_1(t_1, t_2) &= t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} B(p, \tau) c'(p) dp \right]_\tau d\tau \\ &= -\alpha t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha-1} B(p, \tau) c'(p) dp d\tau \\ &\quad + t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} B_\tau(p, \tau) c'(p) dp d\tau \\ &\equiv -\alpha W_{1,1}(t_1, t_2) + W_{1,2}(t_1, t_2). \end{aligned}$$

After having introduced the following functions,

$$Q_{1,1}(t_1, t_2) = t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha-1} [s(\tau) - s(p)] p^{\alpha-1} dp d\tau, \quad (2.31)$$

$$Q_{1,2}(t_1, t_2) = t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \tau^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} p^{\alpha-1} dp d\tau, \quad (2.32)$$

then using (2.14), (2.19), (2.28) and (1.3) we notice,

$$|W_{1,1}(t_1, t_2)| \leq c_0 \|c\|_{X(T)} Q_{1,1}(t_1, t_2),$$

$$|W_{1,2}(t_1, t_2)| \leq c_0 \|c\|_{X(T)} Q_{1,2}(t_1, t_2).$$

Now, the estimate below follows from Lemma 4.2,

$$\lim_{t_2 \rightarrow t_1} |W_1(t_1, t_2)| \leq \lim_{t_2 \rightarrow t_1} c_0 \|c\|_{X(T)} (Q_{1,1}(t_1, t_2) + Q_{1,2}(t_1, t_2)) = 0.$$

Next, we shall deal with  $W_2$ . While keeping in mind the positivity of  $t_1$  and (2.23), we perform differentiation with respect to  $\tau$ . We obtain,

$$\begin{aligned} W_2(t_1, t_2) &= t_2^{1-\alpha} \int_0^{t_1} [(t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1}] \left[ \int_0^\tau (\tau - p)^{-\alpha} B(p, \tau) c'(p) dp \right]_\tau d\tau \\ &= -\alpha t_2^{1-\alpha} \int_0^{t_1} [(t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1}] \int_0^\tau (\tau - p)^{-\alpha-1} B(p, \tau) c'(p) dp d\tau \\ &\quad + t_2^{1-\alpha} \int_0^{t_1} [(t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1}] \int_0^\tau (\tau - p)^{-\alpha} (B(p, \tau))_\tau c'(p) dp d\tau \\ &\equiv -\alpha W_{2,1}(t_1, t_2) + W_{2,2}(t_1, t_2). \end{aligned}$$

Similarly to the estimates of  $W_1$ , we introduce new functions,

$$Q_{2,1}(t_1, t_2) = t_2^{1-\alpha} \int_0^{t_1} [(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1}] \int_0^\tau (\tau - p)^{-\alpha-1} p^{\alpha-1} [s(\tau) - s(p)] dp d\tau, \quad (2.33)$$

$$Q_{2,2}(t_1, t_2) = t_2^{1-\alpha} \int_0^{t_1} [(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1}] \tau^{\alpha-1} \int_0^\tau (\tau - p)^{-\alpha} p^{\alpha-1} dp d\tau. \quad (2.34)$$

From (2.14), (2.16), (2.28) and (1.3) we deduce that

$$|W_{2,1}(t_1, t_2)| \leq c_0 \|c\|_{X(T)} Q_{2,1}(t_1, t_2),$$

$$|W_{2,2}(t_1, t_2)| \leq c_0 \|c\|_{X(T)} Q_{2,2}(t_1, t_2).$$

Thus, Lemma 4.3 implies

$$\lim_{t_2 \rightarrow t_1^+} W_2(t_1, t_2) = 0.$$

Obviously, we also have

$$\lim_{t_2 \rightarrow t_1^+} W_3(t_1, t_2) = 0,$$

and in this way we proved (2.29).

**Case of  $\mathcal{P}_2$ .** Using (1.3) and the continuity of  $c$  we notice that

$$\mathcal{P}_2(t) = t^{1-\alpha} \frac{\dot{s}(t)}{s(t)} \tilde{D}c(t).$$

Hence,  $\mathcal{P}_2 \in C([0, T]; \mathbb{R}^m)$ .

**Case of  $\mathcal{P}_3$ .** For all  $c \in X(T)$  we notice an obvious bound,

$$|c(p)| \leq \|c\|_{X(T)}. \quad (2.35)$$

We shall show that

$$\mathcal{P}_3 \in C([0, T]; \mathbb{R}^m). \quad (2.36)$$

Similarly to the case of  $\mathcal{P}_1$ , if we keep in mind Lemma 2.1 and Lemma 4.1, then it is enough to prove that

$$t \mapsto t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) c(p) dp \right]_\tau d\tau =: \mathcal{P}_{3,1} \in C([0, T]; \mathbb{R}^m). \quad (2.37)$$

We are going to show that the following difference converges to zero when  $t_2 \rightarrow t_1$ .

$$\begin{aligned} & t_2^{1-\alpha} \int_0^{t_2} (t_2-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) c(p) dp \right]_\tau d\tau \\ & - t_1^{1-\alpha} \int_0^{t_1} (t_1-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) c(p) dp \right]_\tau d\tau \\ = & t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) c(p) dp \right]_\tau d\tau \\ & + t_2^{1-\alpha} \int_0^{t_1} [(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1}] \left[ \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) c(p) dp \right]_\tau d\tau \\ & + (t_2^{1-\alpha} - t_1^{1-\alpha}) \int_0^{t_1} (t_1-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) c(p) dp \right]_\tau d\tau \\ \equiv & W_4(t_1, t_2) + W_5(t_1, t_2) + W_6(t_1, t_2). \end{aligned}$$

If  $t_1 = 0$ , then we have  $W_5(0, t_2) = 0$  and  $W_6(0, t_2) = 0$ , as a result  $\mathcal{P}_{3,1}$  is continuous at zero if and only if  $\lim_{t_2 \rightarrow 0} W_4(0, t_2) = 0$ . This is why we estimate  $W_4$ . We proceed as in the case of  $W_1$  and  $W_2$ . We use (2.24) to perform the differentiation with respect to  $\tau$ . Here is the result,

$$\begin{aligned} W_4(t_1, t_2) &= t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \left[ \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) c(p) dp \right]_\tau d\tau \\ &= -\alpha t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha-1} \tilde{D}(p, \tau) c(p) dp d\tau \\ &\quad + t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}_\tau(p, \tau) c(p) dp d\tau \\ &\equiv -\alpha W_{4,1}(t_1, t_2) + W_{4,2}(t_1, t_2). \end{aligned}$$

From (2.18), (2.19), (2.35) and (1.3) while keeping in mind the definitions of  $Q_{1,1}$  and  $Q_{1,2}$ , we obtain that

$$\begin{aligned} |W_{4,1}(t_1, t_2)| &\leq c_0 \|c\|_{X(T)} Q_{1,1}(t_1, t_2), \\ |W_{4,2}(t_1, t_2)| &\leq c_0 \|c\|_{X(T)} Q_{1,2}(t_1, t_2). \end{aligned}$$

Invoking (4.18) and (4.20) from Lemma 4.2 we get the following estimate

$$\lim_{t_2 \rightarrow t_1} |W_4(t_1, t_2)| \leq \lim_{t_2 \rightarrow t_1} c_0 \|c\|_{X(T)} (Q_{1,1}(t_1, t_2) + Q_{1,2}(t_1, t_2)) = 0.$$

Since  $t_1$  may be zero we see that  $\mathcal{P}_{3,1}$  is continuous at  $t = 0$ .

Now, we turn our attention to  $W_5$ . In this case we have  $t_1 > 0$ . We use (2.24) to perform the differentiation with respect to  $\tau$ , resulting with

$$\begin{aligned} W_5(t_1, t_2) &= t_2^{1-\alpha} \int_0^{t_1} [(t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1}] \left[ \int_0^\tau (\tau - p)^{-\alpha} \tilde{D}(p, \tau) c(p) dp \right]_\tau d\tau \\ &= -\alpha t_2^{1-\alpha} \int_0^{t_1} [(t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1}] \int_0^\tau (\tau - p)^{-\alpha-1} \tilde{D}(p, \tau) c(p) dp d\tau \\ &\quad + t_2^{1-\alpha} \int_0^{t_1} [(t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1}] \int_0^\tau (\tau - p)^{-\alpha} (\tilde{D}(p, \tau))_\tau c(p) dp d\tau \\ &\equiv -\alpha W_{5,1}(t_1, t_2) + W_{5,2}(t_1, t_2). \end{aligned}$$

Making use of (2.18), (2.19), (2.35) and (1.3) we arrive at the following estimates,

$$|W_{5,1}(t_1, t_2)| \leq c_0 \|c\|_{X(T)} Q_{2,1}(t_1, t_2),$$

$$|W_{5,2}(t_1, t_2)| \leq c_0 \|c\|_{X(T)} Q_{2,2}(t_1, t_2).$$

We finish estimating  $W_5$  by invoking Lemma 4.3 to obtain,

$$\lim_{t_2 \rightarrow t_1^+} W_5(t_1, t_2) = 0.$$

Since it is easy to see that

$$\lim_{t_2 \rightarrow t_1^+} W_6(t_1, t_2) = 0,$$

then we conclude the proof of (2.36).

**Case of  $\mathcal{P}_4$ .** Since functions  $s$  and  $c$  are absolutely continuous, then due to (2.3) and the definition of  $Ec$  we deduce that  $Ec \in AC$  too. Since  $P_4(t) = I^\alpha(Ec)$ , then by Lemma 2.1 we deduce that

$$\mathcal{P}_4 \equiv t^{1-\alpha} (P_4 c)'(t) = c_0 t^{1-\alpha} \int_0^t (t - \tau)^{\alpha-1} (E(\tau) c(\tau))' d\tau + c_0 E(0) c(0).$$

We infer from (2.20) and (1.3) that function  $t \mapsto t^{1-\alpha} (Ec)'$  is essentially bounded, hence we may apply Lemma 4.4 with  $f = Ec$  to deduce that the  $\mathcal{P}_4$  is Hölder continuous on  $[0, T]$ , i.e. it has better regularity than required.

We also have to check continuity of  $t \mapsto t^{1-\alpha} (I^\alpha G^\varepsilon)'(t)$ , where function  $G^\varepsilon$  is smooth. This follows from Lemma 4.4 applied to  $G^\varepsilon$ . Thus, we finished the proof of (2.27).

The final step in the proof of  $Pc \in X(T)$  is checking that  $(Pc)(0) = c_m(0)$ . For this purpose we shall check that  $P_i(0) = 0$ ,  $i = 1, \dots, 4$  as well as  $I^\alpha G^\varepsilon(0) = 0$ .

We notice that (2.14) and (2.21) imply

$$\begin{aligned}
|(P_1 c)(t)| &\leq c_0 \|c\|_{X(T)} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} [s(\tau) - s(p)] p^{\alpha-1} dp d\tau \\
&\leq c_0 \|c\|_{X(T)} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau p^{\alpha-1} dp d\tau \\
&= c_0 \|c\|_{X(T)} t^{2\alpha} \xrightarrow[t \rightarrow 0]{} 0.
\end{aligned}$$

In next step we see that we deduce from (1.3) that

$$|(P_2 c)(t)| \leq c_0 \|c\|_{X(T)} t^\alpha \xrightarrow[t \rightarrow 0]{} 0.$$

Estimates (2.18) and (2.21) permit us to bound  $P_3$  as follows,

$$\begin{aligned}
|(P_3 c)(t)| &\leq c_0 \|c\|_{X(T)} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-p)^{-\alpha} [s(\tau) - s(p)] \dot{s}(p) dp d\tau \\
&\leq c_0 \|c\|_{X(T)} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau \dot{s}(p) dp d\tau \\
&\leq c_0 \|c\|_{X(T)} t^\alpha [s(t) - s(0)] \xrightarrow[t \rightarrow 0]{} 0.
\end{aligned}$$

The remaining terms,  $P_4 = I^\alpha(Ec)$  and  $I^\alpha G^\epsilon$  tend to zero as  $t$  goes to 0, because they are fractional integrals of bounded functions. Hence, we conclude that  $(Pc)(0) = c_m(0)$  and Lemma 2.3 is proved.  $\square$

### 2.3 Operator $P$ is a contraction

Here, we show that we can choose  $T$  sufficiently small to make  $P$  a contraction on  $X(T)$ .

**Lemma 2.4.** *There exists  $T_0 \leq T$ , which depends only on  $b, m, \alpha, T$  and  $\|s\|_{X(T)}$  such that if  $T_1 < T_0$ , then  $P : X(T_1) \rightarrow X(T_1)$  is contraction.*

*Proof.* We take a number  $T_1 \leq T$ , any elements  $c_1, c_2 \in X(T_1)$  and their difference  $\bar{c} = c_1 - c_2$ . Of course,  $\bar{c}(0) = 0$  and we have,

$$|\bar{c}'(p)| \leq \|\bar{c}\|_{X(T_1)} p^{\alpha-1}, \quad (2.38)$$

$$|\bar{c}(p)| \leq \|\bar{c}\|_{X(T_1)} p^\alpha. \quad (2.39)$$

Since operators  $P_i$  are linear, it is sufficient to show that

$$\|P_i \bar{c}\|_{X(T_1)} \leq \rho_i(T_1) \|\bar{c}\|_{X(T_1)}, \quad (2.40)$$

where  $\rho_i(T_1)$  go to zero, when  $T_1 \rightarrow 0$ ,  $i = 1, \dots, 4$ . We recall that we denote by  $c_0$  a generic constant depending only on the data specified in the assumptions. We estimate operators  $P_i$  one by one.

**Case of  $P_1$ .** Making use of (2.38), (2.14) and (2.21), we can estimate  $P_1\bar{c}$  in the following way, for  $t \leq T_1$ ,

$$|P_1\bar{c}(t)| \leq c_0\|\bar{c}\|_{X(T_1)} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau p^{\alpha-1} dp d\tau = c_0\|\bar{c}\|_{X(T_1)} t^{2\alpha}.$$

As a result we see,

$$\|P_1\bar{c}\|_{C(0,T_1)} \leq c_0 T_1^\alpha \|\bar{c}\|_{X(T_1)}. \quad (2.41)$$

After an application of Lemma 2.1 and Lemma 2.2, then making use of estimates (2.14), (2.16), (2.38) and assumption (1.3) we arrive at

$$|t^{1-\alpha}(P_1\bar{c})'(t)| \leq c_0\|\bar{c}\|_{X(t)}[Q_{1,1}(0,t) + Q_{1,2}(0,t)].$$

The estimates on  $Q_{1,1}$  and  $Q_{1,2}$  provided by Lemma 4.2, combined with (2.21) give us

$$|t^{1-\alpha}(P_1\bar{c})'(t)| \leq c_0 T_1^\alpha \|\bar{c}\|_{X(T_1)} \quad \text{for } t \in [0, T_1]. \quad (2.42)$$

**Case of  $P_2$ .** Taking into account (2.39) and (1.3), we obtain the following estimate,

$$|(P_2\bar{c})(t)| \leq c_0\|\bar{c}\|_{X(T_1)}[s(t) - s(0)].$$

Hence, (2.21) implies that

$$\|P_2\bar{c}\|_{C(0,T_1)} \leq c_0 T_1^\alpha \|\bar{c}\|_{X(T_1)}. \quad (2.43)$$

Finally, combining (2.39) with assumption (1.3) gives us

$$|t^{1-\alpha}(P_2\bar{c})'(t)| \leq c_0 T_1^\alpha \|\bar{c}\|_{X(T_1)} \quad \text{for all } t \leq T_1. \quad (2.44)$$

**Case of  $P_3$ .** In order to estimate  $P_3$ , we use assumption (1.3)) and the bounds (2.39), (2.18), (2.21). Then, we clearly have

$$|P_3\bar{c}(t)| \leq c_0\|\bar{c}\|_{X(T_1)} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau p^{2\alpha-1} dp d\tau = c_0\|\bar{c}\|_{X(T_1)} t^{3\alpha}, \quad \text{for all } t \leq T_1.$$

Hence,

$$\|P_3\bar{c}\|_{C(0,T_1)} \leq c_0 T_1^\alpha \|\bar{c}\|_{X(T_1)}. \quad (2.45)$$

Analogously to the case of  $P_1$ , using Lemma 2.1 and Lemma 2.2 and next (2.18), (2.19), (2.39) and (1.3), we obtain

$$|t^{1-\alpha}(P_3\bar{c})'(t)| \leq c_0 t^\alpha \|\bar{c}\|_{X(t)}[Q_{1,1}(0,t) + Q_{1,2}(0,t)],$$

where  $Q_{1,i}$  were defined in (2.31) and (2.32).

Hence from (4.19), (4.20) and (2.21), we get

$$|t^{1-\alpha}(P_3\bar{c})'(t)| \leq c_0 T_1^\alpha \|\bar{c}\|_{X(T_1)} \quad \text{for } t \in [0, T_1]. \quad (2.46)$$

**Case of  $P_4$ .** Using (2.39), we have

$$|(P_4 c)(t)| \leq c_0 \int_0^t (t - \tau)^{\alpha-1} E(\tau) \bar{c}(\tau) d\tau \leq c_0 \|\bar{c}\|_{X(T_1)} t^\alpha,$$

hence

$$\|P_4 \bar{c}\|_{C(0, T_1)} \leq c_0 T_1^\alpha \|\bar{c}\|_{X(T_1)}. \quad (2.47)$$

It remains to show the estimate for  $t^{1-\alpha}(P_4 \bar{c})'(t)$ . To that end we will use Lemma 2.1 and the fact that  $\bar{c}(0) = 0$ . Then, from (2.20), (2.38) and (1.3), we get

$$\begin{aligned} |t^{1-\alpha}(P_4 \bar{c})'(t)| &\leq c_0 t^{1-\alpha} \int_0^t (t - \tau)^{\alpha-1} [E(\tau) \bar{c}(\tau)]_\tau d\tau \\ &\leq c_0 \|\bar{c}\|_{X(T_1)} t^{1-\alpha} \int_0^t (t - \tau)^{\alpha-1} \tau^{\alpha-1} d\tau = c_0 t^\alpha \|\bar{c}\|_{X(T_1)}, \end{aligned}$$

so

$$|t^{1-\alpha}(P_4 \bar{c})'(t)| \leq c_0 T_1^\alpha \|\bar{c}\|_{X(T_1)} \quad \text{for } t \in [0, T_1]. \quad (2.48)$$

Finally, from (2.41)-(2.48), we get

$$\|P \bar{c}\|_{X(T_1)} \leq c_0 T_1^\alpha \|\bar{c}\|_{X(T_1)}. \quad (2.49)$$

This finishes the proof.  $\square$

## 2.4 Finding the solution on $[0, T]$

We want to find the solution on the interval  $[0, T]$ , which is given to us. However, due to the non local character of the problem, we cannot use directly the approach applicable to ODE's, but we have to modify it.

**Lemma 2.5.** *Let us suppose that the assumptions of Theorem 1.1 hold, in particular  $T > 0$  is given. Then, there exists a unique function  $c \in X(T)$ , i.e.  $c \in C^1([0, T]; \mathbb{R}^m)$  and  $t^{1-\alpha} c' \in C([0, T]; \mathbb{R}^m)$ , which is a solution of (2.13).*

*Proof.* We know from Lemma 2.3 that for any  $T > 0$  operator  $P$  maps  $X(T)$  into itself. By the Banach fixed point theorem and Lemmas 2.3, 2.4 we obtained the unique solution of (2.13) for  $t \in [0, T_1]$ . We want to extend it for the whole interval  $[0, T]$ . For this purpose, we assume that solution  $c_m$  is given on  $[0, t_*]$  and  $c_m(t_*) \in \mathbb{R}^m$ . We shall see that we can extend this function on  $[0, t_e]$ , where  $t_e > t_*$ . The key observation will be that  $t_e - t_* \geq \delta > 0$ , where  $\delta = \delta(\alpha, m, T, \|s\|_{X(T)})$ .

Let us suppose that for  $t_* > 0$  function  $c_m \in X(t_*)$  is a unique solution to (2.13). We introduce

$$Y_{t_*}(T) = \{u \in X(T) : u|_{[0, t_*]} = c_m\},$$

we notice that for  $c_1, c_2 \in X(T)$  such that  $c_1(t) = c_2(t)$  for  $t \in [0, t_*]$ , after writing  $x_i = c_i|_{[t_*, T]}$ ,  $i = 1, 2$ , we have

$$\|c_1 - c_2\|_{X(T)} \leq \max\{1, T^{1-\alpha}\} \|x_1 - x_2\|_{C^1([t_*, T])}$$



and

$$\|x_1 - x_2\|_{C^1([t_*, T])} \leq \max\{1, t_*^{\alpha-1}\} \|c_1 - c_2\|_{X(T)}.$$

As a result, the following metrics on  $Y_{t_*}(T)$

$$\rho_1(c_1, c_2) = \|c_1 - c_2\|_{X(T)} \quad \text{and} \quad \rho_2(c_1, c_2) = \|c_1 - c_2\|_{C^1([t_*, T])}$$

are equivalent. It is of our advantage to rewrite the fixed point problem (2.26) in  $X(T)$ ,

$$c_m(t) = c_m(0) + (Pc_m)(t) \quad (2.50)$$

as a fixed point problem in  $Y_{t_*}(T)$ . For the sake of simplicity of notation we write  $y(t) := c_m(t)$  for  $t \in [0, t_*]$  and  $x := c|_{[t_*, T]}$  for  $c \in Y_{t_*}(T)$ . For  $t > t_*$  we can rewrite (2.50) as

$$\begin{aligned} x(t) = c(t) &= c_m(0) + (P(x + y))(t) \\ &= (P^{t_*}x)(t) + (\bar{P}y)(t), \end{aligned}$$

where  $\bar{P}y$  is fixed, because it depends upon history until  $t = t_*$  and operator  $P^{t_*} : Y_{t_*}(T) \rightarrow Y_{t_*}(T)$  is given as,

$$P^{t_*} = P_1^{t_*} + P_2^{t_*} + P_3^{t_*} + P_4^{t_*}.$$

Here are the definitions,

$$\begin{aligned} (\bar{P}y)(t) &= c(0) - c_\alpha \int_0^{t_*} (t - \tau)^{\alpha-1} \int_0^\tau (\tau - p)^{-\alpha} B(p, \tau) y'(p) dp d\tau - c_\alpha \int_0^{t_*} \frac{\dot{s}(p)}{s(p)} \hat{D}y(p) dp \\ &\quad - c_\alpha \int_0^{t_*} (t - \tau)^{\alpha-1} \int_0^\tau (\tau - p)^{-\alpha} D(p, \tau) y(p) dp d\tau - c_\alpha \int_0^{t_*} (t - \tau)^{\alpha-1} E(\tau) y(\tau) d\tau \\ &\quad - c_\alpha \int_{t_*}^t (t - \tau)^{\alpha-1} \int_0^{t_*} (\tau - p)^{-\alpha} B(p, \tau) y'(p) dp d\tau \\ &\quad - c_\alpha \int_{t_*}^t (t - \tau)^{\alpha-1} \int_0^{t_*} (\tau - p)^{-\alpha} D(p, \tau) y(p) dp d\tau + \frac{1}{\Gamma(1 - \alpha)} G^\varepsilon(t), \end{aligned}$$

and

$$\begin{aligned} (P_1^{t_*}x)(t) &= -c_\alpha \int_{t_*}^t (t - \tau)^{\alpha-1} \int_{t_*}^\tau (\tau - p)^{-\alpha} B(p, \tau) x'(p) dp d\tau, \\ (P_3^{t_*}x)(t) &= -c_\alpha \int_{t_*}^t (t - \tau)^{\alpha-1} \int_{t_*}^\tau (\tau - p)^{-\alpha} D(p, \tau) x(p) dp d\tau, \\ (P_2^{t_*}x)(t) &= -c_\alpha \int_{t_*}^t \frac{\dot{s}(p)}{s(p)} \hat{D}x(p) dp d\tau, \quad (P_4^{t_*}x)(t) = -c_\alpha \int_{t_*}^t (t - \tau)^{\alpha-1} E(\tau) x(\tau) d\tau. \end{aligned}$$

We shall show that operator  $P^{t_*}$  is a contraction on  $Y_{t_*}(T)$ , hence it has a unique fixed point, provided  $T - t_*$  is small enough. Moreover, we have to prove that the difference  $T - t_*$  can be estimated by a universal constant.

We will estimate the Lipschitz constant of each  $P_i^{t_*}$ ,  $i = 1, \dots, 4$  on  $Y_{t_*}(T)$  separately. For  $c^1, c^2 \in Y_{t_*}(T)$ , we shall write  $x = c^1 - c^2$ , moreover  $\|x\|$  denotes  $\|x\|_{C^1([t_*, T]; \mathbb{R}^m)}$ . We also notice that the definition of  $Y_{t_*}(T)$  implies

$$x(t_*) = 0, \quad |x(t)| \leq \|x\|(t - t_*), \quad (2.51)$$

**Case of  $P_1^{t_*}$ .** Using an argument similar to that in the proof of Lemma 2.2, we deduce that function  $g_3(\tau) = \int_{t_*}^{\tau} (\tau - p)^{-\alpha} B(p, \tau) x'(p) dp$  is absolutely continuous. Hence by Lemma 2.1 and (2.23) we get

$$\begin{aligned} (P_1^{t_*} x)'(t) &= \alpha c_\alpha \int_{t_*}^t (t - \tau)^{\alpha-1} \int_{t_*}^{\tau} (\tau - p)^{-\alpha-1} B(p, \tau) x'(p) dp d\tau \\ &\quad - c_\alpha \int_{t_*}^t (t - \tau)^{\alpha-1} \int_{t_*}^{\tau} (\tau - p)^{-\alpha} B_\tau(p, \tau) x'(p) dp d\tau. \end{aligned}$$

In order to estimate the derivative of  $P_1^{t_*} x$  we need to use (2.14), (2.22), (2.16) and (1.3). As a result, we obtain for  $t_* \geq T_1$ ,

$$\begin{aligned} |(P_1^{t_*} x)'(t)| &\leq c_0 \|x\| \int_{t_*}^t (t - \tau)^{\alpha-1} \int_{t_*}^{\tau} (\tau - p)^{-\alpha} p^{\alpha-1} dp d\tau \\ &\quad + c_0 \|x\| \int_{t_*}^t (t - \tau)^{\alpha-1} \tau^{\alpha-1} \int_{t_*}^{\tau} (\tau - p)^{-\alpha} dp d\tau \\ &\leq c_0 \|x\| \int_{t_*}^t (t - \tau)^{\alpha-1} d\tau + c_0 \|x\| \int_{t_*}^t (t - \tau)^{\alpha-1} \tau^{\alpha-1} (\tau - t_*)^{1-\alpha} d\tau \\ &\leq c_0 \|x\| (t - t_*)^\alpha + c_0 \|x\| T_1^{\alpha-1} (t - t_*), \end{aligned}$$

where  $T_1$  is given in Lemma 2.4. Since  $(P_1^{t_*} x)(t_*) = 0$  we arrive at

$$\|P_1^{t_*} x\| \leq c_0 \|x\| (T - t_*)^\alpha. \quad (2.52)$$

**Case of  $P_2^{t_*}$ .** Here, the estimates are simpler. From (2.51) we get

$$|(P_2^{t_*} x)'(t)| = c_0 \frac{\dot{s}(t)}{s(t)} |\hat{D}x(t)| \leq c_0 T_1^{\alpha-1} |x(t)| \leq c_0 T_1^{\alpha-1} \|x\| (t - t_*).$$

Thus,

$$\|\tilde{P}_2 x\| \leq c_0 \|x\| (T - t_*)^\alpha. \quad (2.53)$$

**Case of  $P_3^{t_*}$ .** Arguing as in the proof of Lemma 4.1, we deduce that function  $g_4(\tau) = \int_{t_*}^{\tau} (\tau - p)^{-\alpha} \tilde{D}(p, \tau) x(p) dp$  is absolutely continuous. Then,

$$\begin{aligned} (P_3^{t_*} x)'(t) &= \alpha c_\alpha \int_{t_*}^t (t - \tau)^{\alpha-1} \int_{t_*}^{\tau} (\tau - p)^{-\alpha-1} \tilde{D}(p, \tau) x(p) dp d\tau \\ &\quad - c_\alpha \int_{t_*}^t (t - \tau)^{\alpha-1} \int_{t_*}^{\tau} (\tau - p)^{-\alpha} \tilde{D}_\tau(p, \tau) x(p) dp d\tau, \end{aligned}$$

where we used Lemma 2.1 and (2.24). Combining the estimates (2.18), (2.22), (2.19), (1.3) and (2.51), we arrive at

$$\begin{aligned}
|(P_3^{t_*}x)'(t)| &\leq c_0 \int_{t_*}^t (t-\tau)^{\alpha-1} \int_{t_*}^{\tau} (\tau-p)^{-\alpha} \dot{s}(p) p^{\alpha-1} |x(p)| dp d\tau \\
&\quad + c_0 \int_{t_*}^t (t-\tau)^{\alpha-1} \int_{t_*}^{\tau} (\tau-p)^{-\alpha} \dot{s}(p) \dot{s}(\tau) |x(p)| dp d\tau \\
&\leq c_0 \|x\| T_1^{\alpha-1} \int_{t_*}^t (t-\tau)^{\alpha-1} \int_{t_*}^{\tau} (\tau-p)^{-\alpha} p^{\alpha-1} (p-t_*) dp d\tau \\
&\leq c_0 \|x\| T_1^{\alpha-1} (t-t_*)^{1+\alpha}.
\end{aligned}$$

Hence,

$$\|P_3^{t_*}x\| \leq c_0 \|x\| (T-t_*)^{\alpha}. \quad (2.54)$$

**Case of  $P_4^{t_*}$ .** Using an analog of Lemma 2.1 and  $x(t_*) = 0$ , we get

$$|(P_4^{t_*}x)'(t)| \leq \int_{t_*}^t (t-\tau)^{\alpha-1} |E'(\tau)| |x(\tau)| d\tau + \int_{t_*}^t (t-\tau)^{\alpha-1} |E(\tau)| |x'(\tau)| d\tau.$$

Hence,

$$\begin{aligned}
|(P_4^{t_*}x)'(t)| &\leq c_0 \|x\| \int_{t_*}^t (t-\tau)^{\alpha-1} \tau^{\alpha-1} (\tau-t_*) d\tau + c_0 \|x\| (t-t_*)^{\alpha} \\
&\leq c_0 \|x\| T_1^{\alpha-1} (t-t_*)^{1+\alpha} + c_0 \|x\| (t-t_*)^{\alpha},
\end{aligned}$$

where we applied (2.20), (2.51) and (1.3). Thus,

$$\|P_4^{t_*}x\| \leq c_0 \|x\| (T-t_*)^{\alpha}. \quad (2.55)$$

Finally, from (2.52)-(2.55), we have

$$\|P^{t_*}x\| \leq c_0 \|x\| (T-t_*)^{\alpha},$$

and the proof is finished.  $\square$

*Proof of Theorem 2.1.* Lemma 2.5 yields  $c_m \in AC([0, T]; \mathbb{R}^m)$ , a fixed point of  $P$  in  $X(T)$ , where  $T > 0$  was given in advance, i.e. (2.13) holds. We notice that  $c_m$  satisfies the assumptions of Lemma 4.4, thus  $D^{\alpha}c_m \in C_{loc}^{0,1-\alpha}((0, T]; \mathbb{R}^m)$ . Thus, if we apply Caputo derivative  $D^{\alpha}$  to the both sides of (2.13), we obtain that  $v_m$  defined by (2.1) satisfies the system (2.6), (2.7). Finally, we have  $v_{m,t} = \sum_{n=0}^m c'_{n,m}(t) \varphi_n(x, t) + \sum_{n=0}^m c_{n,m}(t) \varphi_{n,t}(x, t)$ . By Lemma 2.5 we obtain the continuity of  $t^{1-\alpha}c'_m$ . On the other hand, (1.3) implies the continuity of  $t^{1-\alpha}\varphi_{n,t}(x, t)$  in  $\overline{Q_{s,T}}$ , hence  $t^{1-\alpha}v_{m,t}$  is continuous on  $\overline{Q_{s,T}}$ .  $\square$

### 3 Existence of weak solutions

In order to deduce appropriate estimates for  $v_m$ , we need to state a technical lemma.

**Lemma 3.1.** *We assume that  $s$  satisfies (1.3),  $b$  is positive and  $\alpha \in (0, 1)$ . Let us suppose that  $u$  is continuous on  $\overline{Q_{s,T}}$  and such that  $t^{1-\alpha}u_t \in C(\overline{Q_{s,T}})$ . Then, for each  $t \in (0, T]$  the following estimate holds,*

$$\begin{aligned} & D^\alpha \|u(\cdot, t)\|_{L^2(0, s(t))}^2 + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \int_0^b |u(x, t) - u(x, 0)|^2 dx \\ & + \frac{1}{\Gamma(1-\alpha)} \int_b^{s(t)} [t - s^{-1}(x)]^{-\alpha} |u(x, t) - u(x, s^{-1}(x))|^2 dx \\ & \leq 2 \int_0^{s(t)} D_s^\alpha u(x, t) \cdot u(x, t) dx + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - \tau)^{-\alpha} |u(s(\tau), \tau)|^2 \dot{s}(\tau) d\tau. \end{aligned} \quad (3.1)$$

*Proof.* We recall that  $D_s^\alpha$  was defined in (1.2). We can deduce from our assumptions that function  $\xi(t) = \|u(\cdot, t)\|_{L^2(0, s(t))}^2$  is absolutely continuous on  $[0, T]$ , thus the first term on the left-hand-side of (3.1) is defined for a.e.  $t \in (0, T)$ . Furthermore,  $\xi$  satisfies the assumptions of Lemma 4.4, thus  $D_s^\alpha \xi \in C_{loc}^{0, 1-\alpha}((0, T])$ , so both functions are defined for each  $t \in (0, T]$ . This is why we can write

$$\begin{aligned} & \Gamma(1-\alpha) \int_0^{s(t)} D_s^\alpha u(x, t) \cdot u(x, t) dx + \frac{1}{2} \int_0^t (t - \tau)^{-\alpha} |u(s(\tau), \tau)|^2 \dot{s}(\tau) d\tau \\ & - \frac{\Gamma(1-\alpha)}{2} D^\alpha \|u(\cdot, t)\|_{L^2(0, s(t))}^2 \\ & = \int_0^t \int_0^{s(\tau)} (t - \tau)^{-\alpha} u_\tau(x, \tau) u(x, t) dx d\tau + \frac{1}{2} \int_0^t (t - \tau)^{-\alpha} |u(s(\tau), \tau)|^2 \dot{s}(\tau) d\tau \\ & - \frac{1}{2} \int_0^t (t - \tau)^{-\alpha} \frac{d}{d\tau} \int_0^{s(\tau)} |u(x, \tau)|^2 dx d\tau \\ & = \int_0^t \int_0^{s(\tau)} (t - \tau)^{-\alpha} u_\tau(x, \tau) u(x, t) dx d\tau - \int_0^t (t - \tau)^{-\alpha} \int_0^{s(\tau)} u(x, \tau) u_\tau(x, \tau) dx d\tau \\ & = \int_0^t \int_0^{s(\tau)} (t - \tau)^{-\alpha} u_\tau(x, \tau) [u(x, t) - u(x, \tau)] dx d\tau \\ & = \left( \int_0^{t-h} \int_0^{s(\tau)} + \int_{t-h}^t \int_0^{s(\tau)} \right) (t - \tau)^{-\alpha} u_\tau(x, \tau) [u(x, t) - u(x, \tau)] dx d\tau \\ & \equiv I_h^1 + I_h^2, \end{aligned}$$

where  $h > 0$ . Careful integration by parts gives us,

$$\begin{aligned}
I_h^1 &= -\frac{1}{2} \int_0^{t-h} \int_0^{s(\tau)} (t-\tau)^{-\alpha} (|u(x,t) - u(x,\tau)|^2)_\tau dx d\tau \\
&= -\frac{1}{2} \left( \int_0^b \int_0^{t-h} + \int_b^{s(t-h)} \int_{s^{-1}(x)}^{t-h} \right) (t-\tau)^{-\alpha} (|u(x,t) - u(x,\tau)|^2)_\tau d\tau dx \\
&= \frac{\alpha}{2} \left( \int_0^b \int_0^{t-h} + \int_b^{s(t-h)} \int_{s^{-1}(x)}^{t-h} \right) (t-\tau)^{-\alpha-1} |u(x,t) - u(x,\tau)|^2 d\tau dx \\
&\quad - \frac{1}{2} \int_0^b (t-\tau)^{-\alpha} |u(x,t) - u(x,\tau)|^2 dx \Big|_{\tau=0}^{\tau=t-h} \\
&\quad - \frac{1}{2} \int_b^{s(t-h)} (t-\tau)^{-\alpha} |u(x,t) - u(x,\tau)|^2 dx \Big|_{\tau=s^{-1}(x)}^{\tau=t-h} \\
&= \frac{\alpha}{2} \int_0^{t-h} \int_0^{s(\tau)} (t-\tau)^{-\alpha-1} |u(x,t) - u(x,\tau)|^2 dx d\tau \\
&\quad + \frac{1}{2} \int_0^b t^{-\alpha} |u(x,t) - u(x,0)|^2 dx + \frac{1}{2} \int_b^{s(t-h)} (t-s^{-1}(x))^{-\alpha} |u(x,t) - u(x,s^{-1}(x))|^2 dx \\
&\quad - \frac{1}{2} h^{-\alpha} \int_0^{s(t-h)} |u(x,t) - u(x,t-h)|^2 dx.
\end{aligned}$$

We denote the last term by  $I_h^3$ . Therefore, the proof of (3.1) will be finished if we show that  $I_h^2$  and  $I_h^3$  have limits equal to zero, when  $h \rightarrow 0$ . In order to prove this we assume that  $h \leq \frac{1}{2}t$ . Then, from (1.3) we obtain

$$\begin{aligned}
|I_h^2| &\leq 2\|u\|_{C(Q_{s,T})} \|t^{1-\alpha} u_t\|_{C(Q_{s,T})} \int_{t-h}^t \int_0^{s(\tau)} (t-\tau)^{-\alpha} \tau^{\alpha-1} dx d\tau \\
&= 2s(T) \|u\|_{C(Q_{s,T})} \|t^{1-\alpha} u_t\|_{C(Q_{s,T})} \int_{t-h}^t (t-\tau)^{-\alpha} \tau^{\alpha-1} d\tau \\
&\leq 2s(T) \|u\|_{C(Q_{s,T})} \|t^{1-\alpha} u_t\|_{C(Q_{s,T})} \left(\frac{t}{2}\right)^{\alpha-1} \frac{h^{1-\alpha}}{1-\alpha} \xrightarrow{h \rightarrow 0} 0.
\end{aligned}$$

Similarly, we get

$$|I_h^3| \leq h^{2-\alpha} (t-h)^{2\alpha-2} \|t^{1-\alpha} u_t\|_{C(Q_{s,T})}^2 s(t-h) \leq h^{2-\alpha} \left(\frac{t}{2}\right)^{2\alpha-2} \|t^{1-\alpha} u_t\|_{C(Q_{s,T})}^2 s(t) \xrightarrow{h \rightarrow 0} 0,$$

hence the proof is finished.  $\square$

**Lemma 3.2.** *Let us suppose that the assumptions of Theorem 1.1 are satisfied. Then, for each  $m \in \mathbb{N}$ , and  $t \in (0, T]$  the approximate solutions  $v_m$ , given in Theorem 2.1, satisfy*

the following estimate

$$\begin{aligned}
& I^{1-\alpha} \|v_m(\cdot, t)\|_{L^2(0, s(t))}^2 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \tau^{-\alpha} \int_0^b |v_m(x, \tau) - v_m(x, 0)|^2 dx d\tau \\
& + \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_b^{s(\tau)} [\tau - s^{-1}(x)]^{-\alpha} |v_m(x, \tau)|^2 dx d\tau + \int_0^t \int_0^{s(\tau)} |v_{m,x}(x, \tau)|^2 dx d\tau \\
& \leq \frac{s(T)^2}{2} \|g\|_{L^2(Q_{s,t})}^2 + \frac{t^{1-\alpha}}{\Gamma(\alpha)(1-\alpha)} \|v_0\|_{L^2(0,b)}^2 + \delta(\varepsilon),
\end{aligned} \tag{3.2}$$

where  $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$ .

*Proof.* We multiply (2.6) by  $c_{k,m}(t)$  and sum it up over  $k$  from 0 to  $m$

$$\int_0^{s(t)} D_s^\alpha v_m(x, t) \cdot v_m(x, t) dx + \int_0^{s(t)} |v_{m,x}(x, t)|^2 dx = \int_0^{s(t)} g_m^\varepsilon(x, t) v_m(x, t) dx.$$

Applying Theorem 2.1, we deduce that  $v_m$  satisfies the hypothesis of Lemma 3.1. Hence, from this Lemma and the fact that  $v_m(s(t), t) = 0$  for  $t \in [0, T]$ , we get

$$\begin{aligned}
& \frac{1}{2} D^\alpha \|v_m(\cdot, t)\|_{L^2(0, s(t))}^2 + \frac{1}{\Gamma(1-\alpha)} \frac{t^{-\alpha}}{2} \int_0^b |v_m(x, t) - v_m(x, 0)|^2 dx \\
& + \frac{1}{\Gamma(1-\alpha)} \frac{1}{2} \int_b^{s(t)} [t - s^{-1}(x)]^{-\alpha} |v_m(x, t)|^2 dx + \int_0^{s(t)} |v_{m,x}(x, t)|^2 dx \\
& \leq \int_0^{s(t)} g_m^\varepsilon(x, t) v_m(x, t) dx.
\end{aligned} \tag{3.3}$$

Using Young's inequality, we have

$$\begin{aligned}
& \int_0^t \int_0^{s(\tau)} g_m^\varepsilon(x, \tau) v_m(x, \tau) dx d\tau \\
& \leq \frac{s(T)^2}{4} \int_0^t \int_0^{s(\tau)} |g_m^\varepsilon(x, \tau)|^2 dx d\tau + s(T)^{-2} \int_0^t \int_0^{s(\tau)} |v_m(x, \tau)|^2 dx d\tau \\
& \leq \frac{s(T)^2}{4} \int_0^{t+\varepsilon} \int_0^{s(\tau)} |g(x, \tau)|^2 dx d\tau + \frac{1}{2} \int_0^t \int_0^{s(\tau)} |v_{m,x}(x, \tau)|^2 dx d\tau, \\
& \leq \frac{s(T)^2}{4} \|g\|_{L^2(Q_{s,t})}^2 + \frac{1}{2} \int_0^t \int_0^{s(\tau)} |v_{m,x}(x, \tau)|^2 dx d\tau + \delta(\varepsilon),
\end{aligned}$$

where  $\delta(\varepsilon) = \frac{s(T)^2}{4} \sup_{t>0} \int_t^{t+\varepsilon} \int_0^{s(\tau)} |g(x, \tau)|^2 dx d\tau$ . Due to the absolute continuity of the Lebesgue integral we notice that  $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$ . Next, we integrate (3.3) with respect to  $t$  while applying (4.1) and (4.4) to the first term. We use the above inequality to estimate the right-hand-side.  $\square$

Finally, after these preparations, we are ready to prove the main result, Theorem 1.1.

*Proof of Theorem 1.1.* By the assumption  $h \in C^1([0, T])$ , so  $D^\alpha h \in C([0, T])$ , hence  $u$  is a weak solution of (1.1) if and only if  $v$  is a weak solution of (1.7). Here,  $g \in L^2(Q_{s,T})$  and  $v_0 \in L^2(0, b)$  were defined in (1.8). Therefore, we have to show that (1.9) holds.

For  $\varepsilon = 1/m$  Theorem 2.1 yields  $v_m$  satisfying (2.6). We multiply this equality by a test function  $\psi \in C^1([0, T])$  such that  $\psi(T) = 0$ . Then, we get

$$\begin{aligned} \int_0^T \int_0^{s(t)} D_s^\alpha v_m(x, t) \cdot \varphi_k(x, t) dx \psi(t) dt + \int_0^T \int_0^{s(t)} v_{m,x}(x, t) \cdot \varphi_{k,x}(x, t) dx \psi(t) dt \\ = \int_0^T \int_0^{s(t)} g_m^\varepsilon(x, t) \varphi_k(x, t) dx \psi(t) dt. \end{aligned} \quad (3.4)$$

We note that  $v_m$  is absolutely continuous with respect to  $t$ , so is  $I^{1-\alpha} v_m$ . Thus, we may integrate by parts to get

$$\begin{aligned} & \int_0^T \int_0^{s(t)} D_s^\alpha v_m(x, t) \cdot \varphi_k(x, t) dx \psi(t) dt \\ &= \int_0^b \int_0^T \frac{d}{dt} I^{1-\alpha} [v_m(x, t) - v_m(x, 0)] \cdot \varphi_k(x, t) \psi(t) dt dx \\ & \quad + \int_b^{s(T)} \int_{s^{-1}(x)}^T \frac{d}{dt} I^{1-\alpha} v_m(x, t) \cdot \varphi_k(x, t) \psi(t) dt dx \\ &= - \int_0^b \int_0^T I^{1-\alpha} [v_m(x, t) - v_m(x, 0)] \cdot [\varphi_k(x, t) \psi(t)]_t dt dx \\ & \quad - \int_b^{s(T)} \int_{s^{-1}(x)}^T I^{1-\alpha} v_m(x, t) \cdot [\varphi_k(x, t) \psi(t)]_t dt dx \\ &= - \int_0^T \int_0^{s(t)} I_s^{1-\alpha} [v_m(x, t) - \tilde{v}_m(x, 0)] \cdot [\varphi_k(x, t) \psi(t)]_t dx dt, \end{aligned}$$

where we used the fact that  $v_m(x, s^{-1}(x)) = 0$ .

We shall show the following estimate

$$\|I^{1-\alpha} \tilde{v}_m(\cdot, t)\|_{L^2(0, s(T))} \leq I^{1-\alpha} \|v_m(\cdot, t)\|_{L^2(0, s(t))}^2 + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}, \quad \text{for } t \in [0, T], \quad (3.5)$$

where the tilde over  $v_m$  denotes an extension by zero. Indeed, denoting by  $\|\cdot\|$  the norm  $\|\cdot\|_{L^2(0, s(T))}$ , we can write

$$\begin{aligned} \|I^{1-\alpha} \tilde{v}_m(\cdot, t)\| &= \sup_{\|\eta\|=1} \left| \int_0^{s(T)} I^{1-\alpha} \tilde{v}_m(x, t) \eta(x) dx \right| \\ &\leq \frac{1}{\Gamma(1-\alpha)} \sup_{\|\eta\|=1} \int_0^t (t-\tau)^{-\alpha} \int_0^{s(T)} |\tilde{v}_m(x, \tau) \eta(x)| dx d\tau \\ &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \|\tilde{v}_m(\cdot, \tau)\| d\tau \leq \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} [\|\tilde{v}_m(\cdot, \tau)\|^2 + 1] d\tau \\ &= I^{1-\alpha} \|\tilde{v}_m(\cdot, t)\|^2 + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}, \end{aligned}$$

which leads to (3.5).

By Lemma 3.2, we get the estimates of  $\{v_m\}, \{v_{m,x}\}$  in the norm  $L^2(Q_{s,T})$  and by (3.5) the bound for the norm of  $I^{1-\alpha}\tilde{v}_m$  in  $L^\infty(0, T; L^2(0, s(T)))$ . As a result, there exists a subsequence (denoted again by  $\{v_m\}$ ) and  $q \in L^\infty(0, T; L^2(0, s(T)))$  such that  $v_m \rightarrow v$ ,  $v_{m,x} \rightharpoonup v_x$  in  $L^2(Q_{s,T})$  and  $I^{1-\alpha}\tilde{v}_m \xrightarrow{*} q$  in  $L^\infty(0, T; L^2(0, s(T)))$ . Then  $\tilde{v}_m \rightarrow \tilde{v}$  in  $L^2((0, s(T)) \times (0, T))$ . Thus, from inequality

$$\|I^{1-\alpha}w\|_{L^2((0,s(T)) \times (0,T))} \leq c_0\|w\|_{L^2((0,s(T)) \times (0,T))}, \quad (3.6)$$

we obtain  $I^{1-\alpha}\tilde{v}_m \rightarrow I^{1-\alpha}\tilde{v}$  in  $L^2((0, s(T)) \times (0, T))$ . The uniqueness of the weak limit implies  $I^{1-\alpha}\tilde{v} = q \in L^\infty(0, T; L^2(0, s(T)))$ . From the estimate  $|\varphi_k(x, t)\psi(t)|_t \leq c(b, k)\dot{s}(t)$  and (1.3) we have,  $[\varphi_k(x, t)\psi(t)]_t \in L^1(0, T; L^2(0, s(T)))$ . Hence,

$$\int_0^T \int_0^{s(T)} I^{1-\alpha}\tilde{v}_m(x, t) \cdot [\varphi_k(x, t)\psi(t)]_t dx dt \rightarrow \int_0^T \int_0^{s(T)} I^{1-\alpha}\tilde{v}(x, t) \cdot [\varphi_k(x, t)\psi(t)]_t dx dt.$$

and

$$\int_0^T \int_0^{s(t)} D_s^\alpha v_m(x, t) \cdot \varphi_k(x, t) dx \psi(t) dt \rightarrow - \int_0^T \int_0^{s(T)} I_s^{1-\alpha}[v(x, t) - \tilde{v}_0(x)] \cdot [\varphi_k(x, t)\psi(t)]_t dx dt. \quad (3.7)$$

Therefore, from (3.4) we obtain

$$\begin{aligned} & - \int_0^T \int_0^{s(T)} I_s^{1-\alpha}[v(x, t) - \tilde{v}_0(x)] \cdot [\varphi_k(x, t)\psi(t)]_t dx dt + \int_0^T \int_0^{s(t)} v_x(x, t) \cdot \varphi_{k,x}(x, t) dx \psi(t) dt \\ & = \int_0^T \int_0^{s(t)} g(x, t) \varphi_k(x, t) dx \psi(t) dt, \end{aligned} \quad (3.8)$$

which finishes the proof of theorem 1.1. □

## 4 Appendix

In this section we collect facts, which we use in the previous parts of this paper. We begin with the definition of fractional operators and some simple calculations.

### 4.1 Basic facts about fractional operators

For  $\alpha \in (0, 1)$  we define fractional integration of integrable function  $f$

$$I_{t_0}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t - \tau)^{\alpha-1} f(\tau) d\tau.$$

For simplicity we write  $I^\alpha f(t) = I_0^\alpha f(t)$ . One can check, see [8, formula (2.21)], that

$$I^1 f = I^{1-\alpha}(I^\alpha f). \quad (4.1)$$



For absolutely continuous function  $f$  we define the Caputo fractional derivative,

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \dot{f}(\tau) d\tau.$$

Recall that if  $p < t$ , then we have

$$\int_p^t (t-\tau)^{\alpha-1} (\tau-p)^{-\alpha} d\tau = B(1-\alpha, \alpha), \quad (4.2)$$

where  $B(x, y)$  denotes the Beta function. Furthermore,

$$\Gamma(\alpha)\Gamma(1-\alpha) = B(1-\alpha, \alpha). \quad (4.3)$$

If  $f$  is absolutely continuous, then by Fubini theorem and (4.2) we have,

$$I^\alpha D^\alpha f(t) = f(t) - f(0). \quad (4.4)$$

**Lemma 2.1.** *If  $f$  is absolutely continuous, then  $(I^\alpha f)(t) \in AC[0, T]$  and  $(I^\alpha f)'(t) = I^\alpha f'(t) + t^{\alpha-1} \frac{f(0)}{\Gamma(\alpha)}$ .*

*Proof.* Using the well-known characterization of absolutely continuous functions, we can write,

$$(I^\alpha f)(t) = (I^{\alpha+1} f')(t) + (I^\alpha f(0))(t) = I(I^\alpha f')(t) + c_0 f(0) t^\alpha.$$

After taking the derivative of both sides we arrive at the desired statement.  $\square$

## 4.2 Estimates

We will establish two lemmas, which appears to be essential in the proof of Lemma 2.3.

**Lemma 4.1.** *We assume that  $v \in C([0, T]; \mathbb{R}^m)$  and let  $g_1(\tau) = \int_0^\tau (\tau-p)^{-\alpha} \tilde{D}(p, \tau) v(p) dp$ . Then, there exists  $\tilde{g}_1$ , an absolutely continuous representative of integrable function  $g_1$ .*

*Proof.* It is sufficient to show that  $g_1 \in W^{1,1}(0, T)$ , i.e.

$$\int_0^T \varphi'(\tau) g_1(\tau) d\tau = - \int_0^T \varphi(\tau) h_1(\tau) d\tau \quad \text{for every } \varphi \in C_0^\infty(0, T), \quad (4.5)$$

where

$$\int_0^\tau [(\tau-p)^{-\alpha} \tilde{D}(p, \tau)]_\tau v(p) dp = h_1(\tau) \in L^1(0, T). \quad (4.6)$$

For this purpose, it is enough to show that

$$\tau \mapsto \int_0^\tau (\tau-p)^{-\alpha-1} \tilde{D}(p, \tau) v(p) dp \in L^1(0, T), \quad (4.7)$$

$$\tau \mapsto \int_0^\tau (\tau-p)^{-\alpha} (\tilde{D}(p, \tau))_\tau v(p) dp \in L^1(0, T). \quad (4.8)$$

From (2.18) we have

$$\int_0^T \left| \int_0^\tau (\tau - p)^{-\alpha-1} \tilde{D}(p, \tau) v(p) dp \right| d\tau \leq c_0 \int_0^T \int_0^\tau (\tau - p)^{-\alpha-1} \dot{s}(p) [s(\tau) - s(p)] dp d\tau.$$

Using (2.21) and (2.22), we may write that

$$|s(\tau) - s(p)| \leq c_0 p^{\gamma(\alpha-1)} |\tau - p|^{\gamma+(1-\gamma)\alpha} \quad \text{for } p < \tau, \quad (4.9)$$

where  $\gamma \in (0, 1)$  will be chosen below. Hence, using (1.3), we can estimate the above integral by

$$c_0 \int_0^T \int_0^\tau (\tau - p)^{\gamma(1-\alpha)-1} p^{(\alpha-1)(\gamma+1)} dp d\tau,$$

which is finite, provided that  $0 < \gamma < \frac{\alpha}{1-\alpha}$ . Thus, (4.7) follows. Now, we shall show (4.8). Using (2.19), we get

$$\begin{aligned} \int_0^T \left| \int_0^\tau (\tau - p)^{-\alpha} (\tilde{D}(p, \tau))_\tau v(p) dp \right| d\tau &\leq c_0 \int_0^T \dot{s}(\tau) \int_0^\tau (\tau - p)^{-\alpha} \dot{s}(p) dp d\tau \\ &\leq c_0 \int_0^T \tau^{\alpha-1} \int_0^\tau (\tau - p)^{-\alpha} p^{\alpha-1} dp d\tau < \infty. \end{aligned}$$

In this way we proved (4.8) and (4.6).

We are going to show (4.5). By Fubini Theorem we have,

$$\int_0^T \varphi'(\tau) \int_0^\tau (\tau - p)^{-\alpha} \tilde{D}(p, \tau) v(p) dp d\tau = \int_0^T v(p) \int_p^T \varphi'(\tau) (\tau - p)^{-\alpha} \tilde{D}(p, \tau) d\tau dp. \quad (4.10)$$

In order to integrate by parts, we have to show that for each  $p \in (0, T)$  function  $\Psi_p$ , defined below, is absolutely continuous on  $[0, T]$ .

$$\Psi_p(\tau) = \begin{cases} (\tau - p)^{-\alpha} \tilde{D}(p, \tau) & \text{for } \tau > p, \\ 0 & \text{for } \tau \leq p. \end{cases} \quad (4.11)$$

For this purpose, we will show that  $\Psi_p'(\tau) \in L^1(0, T)$  for each  $p \in (0, T)$ . From (2.18) and (2.22) we infer,

$$\int_p^T |(\tau - p)^{-\alpha-1} \tilde{D}(p, \tau)| d\tau \leq c_0 \dot{s}(p) p^{\alpha-1} \int_p^T (\tau - p)^{-\alpha} d\tau.$$

Using (2.19) and (1.3) we obtain

$$\int_p^T |(\tau - p)^{-\alpha} \tilde{D}(p, \tau)_\tau| d\tau \leq c_0 \dot{s}(p) \int_p^T (\tau - p)^{-\alpha} \tau^{\alpha-1} d\tau.$$

Thus, we conclude that  $\Psi_p'(\tau)$  is integrable function for every fixed  $p \in (0, T)$ . Hence, we may integrate by parts in (4.10) and (4.5) follows. As a result,  $g_1 \in W^{1,1}(0, T)$  and by [4, Theorem 1, Ch. 4.9.1],  $g_1$  is equal a.e. in  $[0, T]$  to an absolutely continuous  $\tilde{g}_1$ .

Finally,  $\tilde{g}_1(0) = 0$ , because of (2.18) and (2.21), for any sequence  $\tau_n \searrow 0$ , we get

$$|\tilde{g}_1(\tau_n)| \leq c_0 \int_0^{\tau_n} (\tau_n - p)^{-\alpha} \dot{s}(p) [s(\tau_n) - s(p)] dp \leq c_0 [s(\tau_n) - s(0)] \longrightarrow 0. \quad (4.12)$$

□

**Lemma 2.2.** *We assume that  $p^{1-\alpha}w(p) \in L^\infty(0, T)$  and let  $g_2(\tau) = \int_0^\tau (\tau - p)^{-\alpha} B(p, \tau) w(p) dp$ . Then,  $g_2 = \tilde{g}_2$  a.e. on  $[0, T]$ , where  $\tilde{g}_2 \in AC[0, T]$  and  $\tilde{g}_2(0) = 0$ .*

*Proof.* As in the previous Lemma, we will show that  $g_2 \in W^{1,1}(0, T)$ , i.e.

$$\int_0^T \varphi'(\tau) g_2(\tau) d\tau = - \int_0^T \varphi(\tau) h_2(\tau) d\tau \quad \text{for } \varphi \in C_0^\infty(0, T), \quad (4.13)$$

where

$$h_2(\tau) = \int_0^\tau [(\tau - p)^{-\alpha} B(p, \tau)]_\tau w(p) dp \in L^1(0, T). \quad (4.14)$$

We will show that

$$\tau \mapsto \int_0^\tau (\tau - p)^{-\alpha-1} B(p, \tau) w(p) dp \in L^1(0, T), \quad (4.15)$$

$$\tau \mapsto \int_0^\tau (\tau - p)^{-\alpha} (B(p, \tau))_\tau w(p) dp \in L^1(0, T). \quad (4.16)$$

From (2.14) and the assumption concerning  $w$ , we have,

$$\int_0^T \left| \int_0^\tau (\tau - p)^{-\alpha-1} B(p, \tau) w(p) dp \right| d\tau \leq c_0 \int_0^T \int_0^\tau (\tau - p)^{-\alpha-1} p^{\alpha-1} [s(\tau) - s(p)] dp d\tau.$$

We apply (4.9) and proceed as in the proof of Lemma 4.1 to deduce that the integral on the right-hand-side above is finite.

We are going to show (4.16). Using (2.16) and the estimate for  $w$  we get,

$$\begin{aligned} \int_0^T \left| \int_0^\tau (\tau - p)^{-\alpha} (B(p, \tau))_\tau w(p) dp \right| d\tau &\leq c_0 \int_0^T \dot{s}(\tau) \int_0^\tau (\tau - p)^{-\alpha} p^{\alpha-1} dp d\tau \\ &= c_0 [s(T) - s(0)] < \infty. \end{aligned}$$

Thus, we are able to deduce (4.16) and (4.14).

It remains to show (4.13). By the Fubini theorem, we have

$$\int_0^T \varphi'(\tau) \int_0^\tau (\tau - p)^{-\alpha} B(p, \tau) dp w(p) d\tau = \int_0^T w(p) \int_p^T \varphi'(\tau) (\tau - p)^{-\alpha} B(p, \tau) d\tau dp.$$

As in the proof of Lemma 4.1, we will show that for each  $p \in (0, T)$  function, defined below,

$$\Phi_p(\tau) = \begin{cases} (\tau - p)^{-\alpha} B(p, \tau) & \text{for } \tau > p \\ 0 & \text{for } \tau \leq p \end{cases} \quad (4.17)$$

has integrable derivative on  $(0, T)$ . Making use of (2.14) and (2.22) we get

$$\int_p^T |(\tau - p)^{-\alpha-1} B(p, \tau)| d\tau \leq c_0 p^{\alpha-1} \int_p^T (\tau - p)^{-\alpha} d\tau.$$

Applying (2.15) and (1.3) we obtain the following estimate

$$\int_p^T |(\tau - p)^{-\alpha} B(p, \tau)_\tau| d\tau \leq c_0 \int_p^T (\tau - p)^{-\alpha} \tau^{\alpha-1} d\tau.$$

Thus, following the argument as in the proof of Lemma 4.1, we obtain the claim of the Lemma 2.2.  $\square$

Now, we will prove estimates, which are crucial for proofs of Lemma 2.4 and Lemma 2.5.

**Lemma 4.2.** *We assume that  $\alpha \in (0, 1)$  and  $s$  satisfies (1.3). If function  $Q_{1,1}$  is defined in (2.31),  $Q_{1,2}$  is given by (2.32), then there exists a constant  $c_0 = c_0(\alpha)$  such that the following inequalities hold,*

$$\lim_{t_2 \rightarrow t_1^+} Q_{1,1}(t_1, t_2) = 0 \quad \text{for } t_1 \geq 0, \quad (4.18)$$

$$|Q_{1,1}(0, t)| \leq c_0[s(t) - s(0)], \quad (4.19)$$

$$|Q_{1,2}(t_1, t_2)| \leq c_0|t_2 - t_1|^\alpha. \quad (4.20)$$

*Proof.* **Case of  $Q_{1,1}$ .** With the the Fubini theorem of can write

$$\begin{aligned} Q_{1,1}(t_1, t_2) &= t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} \int_0^\tau (\tau - p)^{-\alpha-1} \int_p^\tau \dot{s}(q) dq p^{\alpha-1} dp d\tau \\ &= t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} \int_0^\tau \dot{s}(q) \int_0^q (\tau - p)^{-\alpha-1} p^{\alpha-1} dp dq d\tau. \end{aligned}$$

Applying substitution  $\tau = t_2 a$  and then  $p = t_2 a b$  we get

$$\begin{aligned} Q_{1,1}(t_1, t_2) &= t_2 \int_{\frac{t_1}{t_2}}^1 (1 - a)^{\alpha-1} \int_0^{t_2 a} \dot{s}(q) \int_0^q (t_2 a - p)^{-\alpha-1} p^{\alpha-1} dp dq da \\ &= \int_{\frac{t_1}{t_2}}^1 (1 - a)^{\alpha-1} a^{-1} \int_0^{t_2 a} \dot{s}(q) \int_0^{\frac{q}{t_2 a}} (1 - b)^{-\alpha-1} b^{\alpha-1} db dq da \\ &= \int_{\frac{t_1}{t_2}}^1 (1 - a)^{\alpha-1} a^{-1} \int_0^{t_2 a} \dot{s}(q) [\alpha^{-1} (1 - b)^{-\alpha} b^\alpha] \Big|_{b=0}^{b=\frac{q}{t_2 a}} dq da \\ &= \int_{\frac{t_1}{t_2}}^1 (1 - a)^{\alpha-1} a^{-1} \int_0^{t_2 a} q^\alpha \dot{s}(q) (t_2 a - q)^{-\alpha} dq da. \end{aligned}$$

Making use of the Fubini theorem again, we have

$$\begin{aligned}
Q_{1,1}(t_1, t_2) &= \int_0^{t_1} \dot{s}(q) q^\alpha \int_{\frac{t_1}{t_2}}^1 (1-a)^{\alpha-1} a^{-1} (t_2 a - q)^{-\alpha} da dq \\
&\quad + \int_{t_1}^{t_2} \dot{s}(q) q^\alpha \int_{\frac{q}{t_2}}^1 (1-a)^{\alpha-1} a^{-1} (t_2 a - q)^{-\alpha} da dq \\
&\equiv Q_{1,1,1}(t_1, t_2) + Q_{1,1,2}(t_1, t_2).
\end{aligned}$$

Obviously,  $Q_{1,1,1}(0, t_2) = 0$ . Therefore, we have to prove the continuity of  $Q_{1,1,1}(t_1, t_2)$  for  $t_1 > 0$ . After having substituted  $b = \frac{1-a}{a-q/t_2}$  and  $a = (q/t_2)b$ , we get

$$Q_{1,1,1}(t_1, t_2) = \int_0^{t_1} \dot{s}(q) (q/t_2)^\alpha \int_0^{\frac{t_2-t_1}{t_1-q}} \frac{b^{\alpha-1}}{1 + \frac{q}{t_2}b} db dq = \int_0^{t_1} \dot{s}(q) \int_0^{\frac{t_2-t_1}{t_2} \frac{q}{t_1-q}} \frac{a^{\alpha-1}}{1+a} da dq.$$

We shall show that above integral is arbitrary small if  $t_2$  is sufficiently close to  $t_1$ . Indeed, if we fix  $\varepsilon > 0$ , then there exists  $c \in (0, t_1)$  such that

$$\int_c^{t_1} \dot{s}(q) \int_0^{\frac{t_2-t_1}{t_2} \frac{q}{t_1-q}} \frac{a^{\alpha-1}}{1+a} da dq < \frac{\varepsilon}{2},$$

because we are dealing with integrable functions. Then,

$$\begin{aligned}
\int_0^c \dot{s}(q) \int_0^{\frac{t_2-t_1}{t_2} \frac{q}{t_1-q}} \frac{a^{\alpha-1}}{1+a} da dq &\leq \int_0^c \dot{s}(q) \int_0^{\frac{t_2-t_1}{t_2} \frac{c}{t_1-c}} \frac{a^{\alpha-1}}{1+a} da dq \\
&\leq [s(c) - s(0)] \int_0^{\frac{t_2-t_1}{t_2} \frac{c}{t_1-c}} \frac{a^{\alpha-1}}{1+a} da < \frac{\varepsilon}{2},
\end{aligned}$$

if  $t_2$  is sufficiently close to  $t_1$ . Therefore,

$$\lim_{t_2 \rightarrow t_1^+} Q_{1,1,1}(t_1, t_2) = 0.$$

In order to estimate  $Q_{1,1,2}$ , we notice that

$$Q_{1,1,2}(t_1, t_2) = \int_{t_1}^{t_2} \dot{s}(q) G_1\left(\frac{q}{t_2}\right) dq,$$

where

$$G_1(x) = x^\alpha \int_x^1 (1-a)^{\alpha-1} a^{-1} (a-x)^{-\alpha} da.$$

The last integral can be easily evaluated. Substituting  $b = \frac{1-a}{a-x}$  and next  $a = xb$ , we get

$$G_1(x) = x^\alpha \int_0^\infty \frac{b^{\alpha-1}}{1+bx} db = \int_0^\infty \frac{a^{\alpha-1}}{1+a} da = \frac{\pi}{\sin \pi \alpha}. \quad (4.21)$$

Hence,

$$Q_{1,1,2}(t_1, t_2) \leq \sup_{x \in (0,1)} G_1(x) \int_{t_1}^{t_2} \dot{s}(q) = c_0[s(t_2) - s(t_1)].$$

Thus,

$$\lim_{t_2 \rightarrow t_1} Q_{1,1}(t_1, t_2) = \lim_{t_2 \rightarrow t_1} (Q_{1,1,1}(t_1, t_2) + Q_{1,1,2}(t_1, t_2)) = 0$$

and

$$|Q_{1,1}(0, t)| = |Q_{1,1,2}(0, t)| \leq c_0[s(t) - s(0)].$$

Now, we shall deal with  $Q_{1,2}$ . We can write

$$Q_{1,2}(t_1, t_2) = c_0 t_2^{1-\alpha} \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} \tau^{\alpha-1} d\tau.$$

Applying substitution  $\tau = t_2 - (t_2 - t_1)a$ , we get

$$c_0(t_2 - t_1)^\alpha \int_0^1 a^{\alpha-1} \left(1 - \frac{t_2 - t_1}{t_2} a\right)^{\alpha-1} da \leq c_0(t_2 - t_1)^\alpha \int_0^1 a^{\alpha-1} (1 - a)^{\alpha-1} da,$$

and the proof is finished.  $\square$

**Lemma 4.3.** *Let us assume that  $\alpha \in (0, 1)$  and  $s$  satisfies (1.3). If function  $Q_{2,1}$  (resp.  $Q_{2,2}$ ) is defined by (2.33) (resp. by (2.34)), then*

$$\lim_{t_2 \rightarrow t_1^+} Q_{2,i}(t_1, t_2) = 0 \quad \text{for } t_1 > 0, \quad i = 1, 2. \quad (4.22)$$

*Proof.* We first deal with  $Q_{2,1}$ ,

$$Q_{2,1}(t_1, t_2) = t_2^{1-\alpha} \int_0^{t_1} [(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1}] \int_0^\tau (\tau - p)^{-\alpha-1} p^{\alpha-1} \int_p^\tau \dot{s}(q) dq dp d\tau.$$

Applying the Fubini theorem and substitutions  $\tau = t_1 a$  and next  $p = t_1 a b$ , we see that

$$\begin{aligned} Q_{2,1}(t_1, t_2) &= t_2^{1-\alpha} \int_0^{t_1} [(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1}] \int_0^\tau \dot{s}(q) \int_0^q (\tau - p)^{-\alpha-1} p^{\alpha-1} dp dq d\tau \\ &= t_2^{1-\alpha} t_1 \int_0^1 [(t_1 - t_1 a)^{\alpha-1} - (t_2 - t_1 a)^{\alpha-1}] \int_0^{t_1 a} \dot{s}(q) \int_0^q (t_1 a - p)^{-\alpha-1} p^{\alpha-1} dp dq da \\ &= t_2^{1-\alpha} t_1^{\alpha-1} \int_0^1 [(1 - a)^{\alpha-1} - (\frac{t_2}{t_1} - a)^{\alpha-1}] a^{-1} \int_0^{t_1 a} \dot{s}(q) \int_0^{\frac{q}{t_1 a}} (1 - b)^{-\alpha-1} b^{\alpha-1} db dq da \\ &= \alpha \left(\frac{t_2}{t_1}\right)^{1-\alpha} \int_0^1 [(1 - a)^{\alpha-1} - (\frac{t_2}{t_1} - a)^{\alpha-1}] a^{-1} \int_0^{t_1 a} \dot{s}(q) (1 - b)^{-\alpha} b^\alpha \Big|_{b=0}^{b=\frac{q}{t_1 a}} dq da \\ &= \alpha \left(\frac{t_2}{t_1}\right)^{1-\alpha} \int_0^1 [(1 - a)^{\alpha-1} - (\frac{t_2}{t_1} - a)^{\alpha-1}] a^{-1} \int_0^{t_1 a} \dot{s}(q) \left(\frac{t_1 a}{q} - 1\right)^{-\alpha} dq da. \end{aligned}$$

Using again the Fubini theorem, we get,

$$Q_{2,1}(t_1, t_2) = \alpha \left( \frac{t_2}{t_1} \right)^{1-\alpha} \int_0^{t_1} \dot{s}(q) \left( \frac{q}{t_1} \right)^\alpha \int_{\frac{q}{t_1}}^1 [(1-a)^{\alpha-1} - (\frac{t_2}{t_1} - a)^{\alpha-1}] a^{-1} \left( a - \frac{q}{t_1} \right)^{-\alpha} dadq.$$

By (4.21) we arrive at

$$\begin{aligned} Q_{2,1}(t_1, t_2) &= \alpha \int_0^{t_1} \int_{\frac{q}{t_1}}^1 \dot{s}(q) \left( \frac{q}{t_1} \right)^\alpha (1-a)^{\alpha-1} a^{-1} \left( a - \frac{q}{t_1} \right)^{-\alpha} dadq \\ &= \alpha \int_0^{t_1} \dot{s}(q) G_1 \left( \frac{q}{t_1} \right) dq \leq c_0 [s(t_1) - s(0)] < \infty. \end{aligned}$$

Thus, we can apply Lebesgue dominated convergence theorem and we see that

$$\lim_{t_2 \rightarrow t_1^+} Q_{2,1}(t_1, t_2) = 0.$$

Now, we estimate  $Q_{2,2}$ . We see,

$$|Q_{2,2}(t_1, t_2)| = c_0 t_2^{1-\alpha} \int_0^{t_1} [(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1}] \tau^{\alpha-1} d\tau.$$

Thus, applying Lebesgue dominated convergence theorem, we get  $\lim_{t_2 \rightarrow t_1^+} Q_{2,2}(t_1, t_2) = 0$  and the proof is finished.  $\square$

**Lemma 4.4.** *Let us assume that  $\alpha \in (0, 1)$ ,  $f \in AC[0, T]$  and  $t^{1-\alpha} f' \in L^\infty(0, T)$ . Then,*

$$|t_2^{1-\alpha} (I^\alpha f')(t_2) - t_1^{1-\alpha} (I^\alpha f')(t_1)| \leq c_0 \|t^{1-\alpha} f'\|_{L^\infty(0, T)} |t_2 - t_1|^\alpha, \quad (4.23)$$

where  $c_0$  depends only on  $\alpha$ . In particular,  $t \mapsto t^{1-\alpha} (I^\alpha f')(t) \in C^{0, \alpha}([0, T])$  and  $D^\alpha f \in C_{loc}^{0, 1-\alpha}((0, T])$ .

*Proof.* For the sake of the simplicity of notation in this proof, we write  $\|f\| := \|t^{1-\alpha} f'\|_{L^\infty(0, T)}$  and assume that  $t_2 > t_1$ . Then we have

$$\begin{aligned} & \Gamma(\alpha) |t_2^{1-\alpha} I^\alpha f'(t_2) - t_1^{1-\alpha} I^\alpha f'(t_1)| \\ & \leq |t_2^{1-\alpha} - t_1^{1-\alpha}| \left| \int_0^{t_2} (t_2 - \tau)^{\alpha-1} f'(\tau) d\tau \right| \\ & \quad + t_1^{1-\alpha} \left| \int_0^{t_2} (t_2 - \tau)^{\alpha-1} f'(\tau) d\tau - \int_0^{t_1} (t_1 - \tau)^{\alpha-1} f'(\tau) d\tau \right| \\ & \equiv B_1(t_1, t_2) + B_2(t_1, t_2), \end{aligned}$$

and  $B_2(0, t_2) = 0$ . Using the obvious inequality  $|f'(s)| \leq \|f\| s^{\alpha-1}$  and the substitution  $\tau = t_2 s$ , we can write

$$\begin{aligned} B_1(t_1, t_2) &\leq \|f\| |t_2^{1-\alpha} - t_1^{1-\alpha}| \int_0^{t_2} (t_2 - \tau)^{\alpha-1} \tau^{\alpha-1} d\tau \\ &= \|f\| |t_2^{1-\alpha} - t_1^{1-\alpha}| t_2^{2\alpha-1} \int_0^1 (1-s)^{\alpha-1} s^{\alpha-1} ds = c_0 \|f\| (t_2^\alpha - t_1^\alpha) h \left( \frac{t_1}{t_2} \right), \end{aligned}$$

where  $h(x) = \frac{1-x^{1-\alpha}}{1-x^\alpha}$ . Since  $|t_2^\alpha - t_1^\alpha| \leq |t_2 - t_1|^\alpha$  and function  $h(x)$  is bounded on  $[0, 1]$ , we get

$$B_1(t_1, t_2) \leq c_0 \|f\| |t_2 - t_1|^\alpha.$$

Now, we estimate  $B_2$ . In this case  $t_1 > 0$ . We use substitution  $s = t_1 - \tau$  and decompose  $B_2$  as follows,

$$\begin{aligned} & B_2(t_1, t_2) \\ &= t_1^{1-\alpha} \left| \int_0^{t_2} (t_2 - \tau)^{\alpha-1} f'(\tau) d\tau - \int_0^{t_1} (t_1 - \tau)^{\alpha-1} f'(\tau) d\tau \right| \\ &= t_1^{1-\alpha} \left| - \int_{t_1}^{t_1-t_2} ((t_2 - t_1) + s)^{\alpha-1} f'(t_1 - s) ds + \int_{t_1}^0 s^{\alpha-1} f'(t_1 - s) ds \right| \\ &= t_1^{1-\alpha} \left| \int_{t_1-t_2}^0 ((t_2 - t_1) + s)^{\alpha-1} f'(t_1 - s) ds + \int_0^{t_1} [((t_2 - t_1) + s)^{\alpha-1} - s^{\alpha-1}] f'(t_1 - s) ds \right|. \end{aligned}$$

Applying  $|f'(s)| \leq \|f\| s^{\alpha-1}$  we reach the following estimate

$$\begin{aligned} B_2(t_1, t_2) &\leq \|f\| t_1^{1-\alpha} \int_{t_1-t_2}^0 ((t_2 - t_1) + s)^{\alpha-1} (t_1 - s)^{\alpha-1} ds \\ &\quad + \|f\| t_1^{1-\alpha} \int_0^{t_1} [s^{\alpha-1} - ((t_2 - t_1) + s)^{\alpha-1}] (t_1 - s)^{\alpha-1} ds \\ &\equiv \|f\| B_{2,1}(t_1, t_2) + \|f\| B_{2,2}(t_1, t_2). \end{aligned}$$

First, we deal with  $B_{2,2}$ . After substitution  $(t_2 - t_1)t = s$ , we get

$$\begin{aligned} B_{2,2}(t_1, t_2) &= t_1^{1-\alpha} (t_2 - t_1)^{\alpha-1} \int_0^{t_1} \left[ \left( \frac{s}{t_2 - t_1} \right)^{\alpha-1} - \left( 1 + \frac{s}{t_2 - t_1} \right)^{\alpha-1} \right] (t_1 - s)^{\alpha-1} ds \\ &= t_1^{1-\alpha} (t_2 - t_1)^{\alpha-1} \int_0^{\frac{t_1}{t_2-t_1}} [t^{\alpha-1} - (1+t)^{\alpha-1}] (t_1 - (t_2 - t_1)t)^{\alpha-1} (t_2 - t_1) dt \\ &= (t_2 - t_1)^\alpha \int_0^{\frac{t_1}{t_2-t_1}} [t^{\alpha-1} - (1+t)^{\alpha-1}] \left( 1 - \frac{t_2 - t_1}{t_1} t \right)^{\alpha-1} dt. \end{aligned}$$

We write  $a = \frac{t_1}{t_2-t_1} > 0$ . We have to estimate the following integral,

$$g(a) = \int_0^a [t^{\alpha-1} - (1+t)^{\alpha-1}] \left( 1 - \frac{t}{a} \right)^{\alpha-1} dt, \quad (4.24)$$

independently of  $a \in (0, \infty)$ . We have to consider a number of cases. First, we assume that  $a \in (0, 2]$ . Then, using substitution  $as = a - t$ , we get,

$$\begin{aligned} g(a) &= a^{1-\alpha} \int_0^a [t^{\alpha-1} - (1+t)^{\alpha-1}] (a-t)^{\alpha-1} dt \leq a^{1-\alpha} \int_0^a t^{\alpha-1} (a-t)^{\alpha-1} dt \\ &= a^\alpha B(1-\alpha, 1-\alpha) \leq 2^\alpha B(1-\alpha, 1-\alpha). \end{aligned}$$



If  $a \in (2, \infty)$ , we can write

$$g(a) = \left( \int_0^1 + \int_1^{\frac{a}{2}} + \int_{\frac{a}{2}}^a \right) [t^{\alpha-1} - (1+t)^{\alpha-1}] \left(1 - \frac{t}{a}\right)^{\alpha-1} dt.$$

Then, applying  $(1 - \frac{t}{a})^{\alpha-1} \leq (1-t)^{\alpha-1}$ , we get

$$\begin{aligned} \int_0^1 [t^{\alpha-1} - (1+t)^{\alpha-1}] \left(1 - \frac{t}{a}\right)^{\alpha-1} dt &\leq \int_0^1 [t^{\alpha-1} - (1+t)^{\alpha-1}] (1-t)^{\alpha-1} dt \\ &= g(1) \leq B(1-\alpha, 1-\alpha). \end{aligned}$$

Next, by the mean theorem, we obtain

$$t^{\alpha-1} - (1+t)^{\alpha-1} \leq (1-\alpha)t^{\alpha-2}. \quad (4.25)$$

Hence, since  $a-t \geq \frac{a}{2}$ , we obtain

$$\begin{aligned} &\int_1^{\frac{a}{2}} [t^{\alpha-1} - (1+t)^{\alpha-1}] \left(1 - \frac{t}{a}\right)^{\alpha-1} dt \leq (1-\alpha) \int_1^{\frac{a}{2}} t^{\alpha-2} \left(1 - \frac{t}{a}\right)^{\alpha-1} dt \\ &= (1-\alpha)a^{1-\alpha} \int_1^{\frac{a}{2}} t^{\alpha-2} (a-t)^{\alpha-1} dt \leq (1-\alpha)2^{1-\alpha} \int_1^{\frac{a}{2}} t^{\alpha-2} dt = 2^{\alpha-1} [1 - (\frac{a}{2})^{\alpha-1}] \\ &\leq 2^{\alpha-1}. \end{aligned}$$

Using again (4.25), we deduce,

$$\begin{aligned} &\int_{\frac{a}{2}}^a [t^{\alpha-1} - (1+t)^{\alpha-1}] \left(1 - \frac{t}{a}\right)^{\alpha-1} dt \leq (1-\alpha) \int_{\frac{a}{2}}^a t^{\alpha-2} \left(1 - \frac{t}{a}\right)^{\alpha-1} dt \\ &\leq 2^{2-\alpha} a^{\alpha-2} \int_{\frac{a}{2}}^a \left(1 - \frac{t}{a}\right)^{\alpha-1} dt = \frac{4^{1-\alpha}}{\alpha} a^{\alpha-1} \leq \frac{2^{1-\alpha}}{\alpha}. \end{aligned}$$

Therefore,  $\sup_{a \in (0, \infty)} g(a) \leq c_0$  and as a result,

$$B_{2,2}(t_1, t_2) \leq c_0 |t_2 - t_1|^\alpha.$$

Now, we turn to  $B_{2,1}$ . Here,  $s = t_1 - \tau$  is negative, hence from  $(1 - \frac{s}{t_1})^{\alpha-1} \leq 1$  we get

$$\begin{aligned} B_{2,1}(t_1, t_2) &= t_1^{1-\alpha} \int_{t_1-t_2}^0 ((t_2 - t_1) + s)^{\alpha-1} (t_1 - s)^{\alpha-1} ds \\ &= (t_2 - t_1)^{\alpha-1} \int_{t_1-t_2}^0 \left(1 + \frac{s}{t_2 - t_1}\right)^{\alpha-1} \left(1 - \frac{s}{t_1}\right)^{\alpha-1} ds \\ &\leq (t_2 - t_1)^{\alpha-1} \int_{t_1-t_2}^0 \left(1 + \frac{s}{t_2 - t_1}\right)^{\alpha-1} ds = \frac{1}{\alpha} (t_2 - t_1)^\alpha. \end{aligned}$$

Thus,

$$B_2(t_1, t_2) \leq c \|f\| (t_2 - t_1)^\alpha,$$

and the proof of (4.23) is completed. If we set  $t^{1-\alpha}(I^\alpha f')(t)|_{t=0} = 0$ , then (4.23) means that  $t^{1-\alpha}(I^\alpha f')(t) \in C^{0,\alpha}([0, T])$  and then  $(I^\alpha f')(t) \in C_{loc}^{0,\alpha}((0, T])$ . To finish the proof it is enough to notice that  $D^{1-\alpha}f = I^\alpha f'$ .  $\square$

## References

- [1] P.Biler, G.Karch, W.A.Woyczyński, Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **18** (2001), 613–637.
- [2] P.Biler, G.Karch, R.Monneau, Nonlinear diffusion of dislocation density and self-similar solutions, *Comm. Math. Phys.*, **294**, (2010), 145–168.
- [3] K. Diethelm, *The analysis of fractional differential equations*, Lecture Notes in Mathematics, 2004.
- [4] L. Evans, R. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
- [5] E.Foufoula-Georgiou, V.Ganti, W.Dietrich, A nonlocal theory of sediment transport on hillslopes, *J. Geophys. Res.*, **115**, (2010), doi:10.1029/2009JF001280
- [6] A. Kilbas, H.M. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.
- [7] J.Klafter, I.M.Sokolov, First steps in random walks. From tools to applications. Oxford University Press, Oxford, 2011.
- [8] S.Samko, A.Kilbas, O. Marichev, *Fractional integrals and derivatives. Theory and applications*, Edited and with a foreword by S. M. Nikol'skii. Translated from the 1987 Russian original. Revised by the authors. Gordon and Breach Science Publishers, Yverdon, 1993.
- [9] R.Schumer, M.M.Meerschaert, B.Baeumer, Fractional advection-dispersion equations for modeling transport at the Earth surface, *J. Geophys. Res.*, **114**, (2009), doi:10.1029/2008JF001246.
- [10] V.R.Voller, V. Ganti, C.Paola, E.Foufoula-Georgiou, Does the flow of information in a landscape have direction? *Geophys. Res. Lett.*, **39**, (2012), doi:10.1029/2011GL050265.
- [11] V.R. Voller, C. Paola, Can anomalous diffusion describe depositional fluvial profiles? *J. Geophys. Res.*, **115**, (2010), doi:10.1029/2009JF001278.